1. Let

\[ \sigma_2(n) = \sum_{d \mid n} d^2. \]

Using Problem 6 from Assignment 1 and the fact that for \( r \in (1, \infty) \) we have

\[ \sum_{n \leq x} \frac{1}{n^r} = \zeta(r) + O \left( \frac{1}{x^{r-1}} \right), \]

which you don’t have to prove, show that

\[ \sum_{n \leq x} \sigma_2(n) = \frac{\zeta(3)}{3} x^3 + O(x^2). \]

2. As an alternative means of finding the Euler Product for \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \), note that for \( \text{Re}(s) > 1 \),

\[ \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \sum_{k=0}^{\infty} \frac{1}{p^{sk}} = \prod_{p \text{ prime}} \varphi_p(s). \]

where \( \varphi_p(s) = \sum_{k=0}^{\infty} \frac{1}{p^{sk}} \) for \( \text{Re}(s) > 1 \). Ignoring issues of convergence, we can give a formula for \( \varphi_p(s) \) as a rational function in \( p^{-s} \) by using the recursion:

\[ \frac{1}{p^s} \varphi_p(s) = \sum_{k=0}^{\infty} \frac{1}{p^{sk+1}} = \sum_{k=1}^{\infty} \frac{1}{p^{sk}} = \sum_{k=0}^{\infty} \frac{1}{p^{sk}} - 1 = \varphi_p(s) - 1 \]

and then solving the recursion \( p^{-s} \varphi_p(s) = \varphi_p(s) - 1 \) to get \( \varphi_p(s) = (1 - p^{-s})^{-1} \), we have

\( \zeta(s) = \prod_p (1 - p^{-s})^{-1} \) for \( \text{Re}(s) > 1 \).

Now, let \( f : \mathbb{N} \to \mathbb{R} \) (or \( \mathbb{C} \)) be a multiplicative function such that for \( r, p \in \mathbb{N} \) with \( p \) prime we have \( f(p^{r+1}) = f(p)f(p^r) - f(p^r) \). Ignoring issues of convergence, use similar methods to find an Euler Product for \( L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \).
3. Recall the derivative \( \frac{d}{dx} \log(f(x)) = \frac{f'(x)}{f(x)} \) (remember in Number Theory all logarithms are base \( e \)) as well as the Taylor Series expansion

\[- \log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \text{ when } |x| < 1.\]

Using these facts together with the Euler Product of \( \zeta(\sigma) \) give a formula for \( c : \mathbb{N} \to \mathbb{R} \) where \( c(n) \) is such that

\[- \frac{\zeta'(\sigma)}{\zeta(\sigma)} = \sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma} \text{ when } \sigma > 1.\]

Don’t worry about issues of convergence or differentiability, it all checks out.
[Hint: You’ve seen \( c(n) \) before.]

4. Show that for the polynomial \( f(x) = x^5 + 115x^3 + 124x \), \( f(n) \) is divisible by 120 for all \( n \in \mathbb{Z} \). [Hint: Reduce and factor.]

5. For any \( n \) with decimal notation (base 10) of the form \( n = a_m \cdots a_5a_4a_3a_2a_1a_0 \) with \( a_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) and \( a_m \neq 0 \), let

\[s_n = a_0 - 3a_1 - 4a_2 - a_3 + 3a_4 + 4a_5 + \cdots + (-1)^m c_m a_m = \sum_{k=0}^{m} (-1)^k c_k a_k\]

where \( c_k = 1, 3, \) or \(-4\) if \( k \equiv 0, 1 \) or \( 2 \) (mod 3) respectively. Show that \( 13 | n \) iff \( 13 | s_n \).

6. Find all \( x \in \mathbb{Z} \) such that \( 8x \equiv 2 \) (mod 14), \( 21x \equiv 6 \) (mod 51) and \( 6x \equiv 8 \) (mod 22), simultaneously.

7. Show that if \( n > 4 \) is composite then \( (n - 2)! \equiv 0 \) (mod \( n \)).

8. Let \( a \in \mathbb{Z} \) with \( (a, 9139) = 1 \), show that \( a^{1332} \equiv 1 \) (mod 9139). [Hint: Factor 9139.]

9. Find all \( x \in \mathbb{Z} \) such that the set \( \{ \frac{1^5}{5} + \frac{n^3}{13} + \frac{x^n}{65} \mid n \in \mathbb{Z} \} \) is a set of integers.

10. Let \( p \) be a prime greater than 3, show that the numerator of

\[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1},\]

when written in lowest terms, is divisible by \( p^2 \).
[Hint: Use that \( \frac{1}{x} + \frac{1}{p-x} = \frac{x}{x(p-1)} \), and also Wilson’s Theorem. It may also be helpful to recall Problem 6 in Assignment 1, or the Proof of Euler’s Theorem.]