Recall: $A = UV^*$ where $U$ is orthogonal, $V$ is orthogonal, and $S$ is diagonal (eigenvalues of $A^*A$).

We use $U$, $V$ as orthonormal bases to write $A$ in diagonal form.

Proposition: $A = \mathbb{E}_{\text{svd}}$ where $\mathbb{E}_{\text{svd}}(A) = UDV^*$.

Start with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Here pool $U$ is $\mathbb{E}_{\text{svd}}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

To get $U$, we take $U = \mathbb{E}_{\text{svd}}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Special Value Decomposition: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The special value decomposition of $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The special value decomposition (SVD) of $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Non-zero singular values:
\[ \text{If } j \leq r, \quad \tilde{A}v_j = \sigma_j v_j \quad \text{and} \quad \text{if } j > r, \quad \tilde{A}v_j = 0 \quad \text{and} \quad \sum_{k=1}^{r} \sigma_k v_k^* v_j = 0 \]

\[ \text{Lemma: If } A \text{ can be represented as } A = \tilde{U} \Sigma \tilde{V}^* \quad \text{where } \delta_k > 0 \]

\[ \tilde{U}, \tilde{V}, \Sigma \text{ are orthonormal} \]

Then this is also an SVD.

**Proof:** Check that \( \tilde{v}_1, \tilde{v}_2 \) are the eigenvalues of \( A^* A \) \( \text{with nonzero eigenvalue}, \sigma_k^2 > 0 \).

Also \( \text{max} \frac{1}{\sigma_k^2} \text{ or entries of } \tilde{A}^* \tilde{A} \).

**Corollary:** If \( A = \sum_{k=1}^{r} \sigma_k v_k^* u_k \) is an SVD, then \( A^* = \sum_{k=1}^{r} \sigma_k u_k^* v_k \) is also an SVD.

\( A, A^* \) have the same nonzero singular values.

**Proof:** From Lemma.

**Aside:** \( |A^*| = |A| \iff |A^*| \neq |A| \)

If \( A = \sum_{k=1}^{r} \sigma_k v_k^* u_k \) then can be written as the product of matrices

\[ A = \tilde{W} \Sigma \tilde{V}^* \]

where \( \tilde{W} = [\tilde{w}_1, \ldots, \tilde{w}_r] \quad m \times r \)

\[ \Sigma = \text{diag} \sigma_1, \sigma_2, \ldots, \sigma_r \quad r \times r \]

\[ \tilde{V} = [\tilde{v}_1, \ldots, \tilde{v}_r] \quad n \times r \]
So complete $\tilde{v}_1, \ldots, \tilde{v}_r$ to a basis $\tilde{v}_1, \ldots, \tilde{v}_m$, and complete $\tilde{w}_1, \ldots, \tilde{w}_r$ to a basis $\tilde{w}_1, \ldots, \tilde{w}_n$. 

and let $\Sigma$ be constructed by padding the bottom-right of $\Sigma$ with zeroes to make an $m \times n$ matrix. 

Henceforth, such a matrix is diagonalisable.

Then: $A = W \Sigma V^*$

where $W = [\tilde{w}_1, \ldots, \tilde{w}_m]$ unitary matrix

$\Sigma = \text{diag } \epsilon_1, \ldots, \epsilon_m$ diagonalisation matrix

$V = [\tilde{v}_1, \ldots, \tilde{v}_n]$ unitary matrix

§6.4 Why the @! #? & we care about SVD:

- Let $A$ be a diagonal matrix with diagonal entries $s_1, \ldots, s_n > 0$.

Let's look at the image of the unit ball $B = \{ x \in \mathbb{R}^n : \| x \| = 1 \}$.

Let $d_B = \{ x \in \mathbb{R}^n : \| x \| = 1 \}$ boundary of $B$.

If $x \in d_B$ let $x = (x_1) \ldots (x_n)$ then $\sum_{k=1}^{n} x_k^2 \leq 1$ i.e. $x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1$.

Let $y = (y_1) \ldots (y_n) = A x = (s_1 x_1) \ldots (s_n x_n)$.

Since $y_k = x_k \Rightarrow (y_1)^2 + (y_2)^2 + \ldots + (y_n)^2 \leq 1$

this is an ellipse - type thing: an $n$-dimensional ellipsoid with semi-axes of length $s_k$ and principle axes $\tilde{e}_1, \ldots, \tilde{e}_n$.

This to notice: works for SVD, since $W, V^*$ are unitary.

$\Sigma$ is diagonalisable.
Saying that \( A = W \Sigma V^* \) for \( w, v \) unitary and diagonal is diagonal.

Says that \( w \) with respect to the bases

\[
A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}, \quad B = \begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_n \end{bmatrix}
\]

\[ [A]_{B,A} \text{ is diagonal} \]

Let \( A \) be any \( mxn \) matrix, \( B \) the unit ball in \( \mathbb{R}^n \).

Then \( A(B) \) is an ellipsoid in \( \text{ran} A \) with axes \( \alpha_1, \alpha_2, \ldots, \alpha_r \) and principal axes \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r \) if \( r \) is the number of nonzero singular values.

Let \( S \) be the largest singular value.

Then \( \| Ax \| \leq S \| x \| \) for all \( x \in \mathbb{R}^n \).

\[
\max_{\| x \| = 1} \| Ax \| = S
\]

Call \( \| A \| = S \), the operator norm of \( A \).

There's another norm: \( \| A \|_2 = \sqrt{\text{trace} (A^* A)} = \sqrt{\sum_{i=1}^{n} \alpha_i^2} \geq \| A \| \)

Also called Frobenius norm.

Let \( A \) be invertible and so the smallest singular value.

Then the condition number is \( \| A \| \| A^{-1} \| = \frac{S}{S_n} \)

If \( Ax = b \)

\[
A \Delta x = \Delta b
\]

Then \( \| \Delta x \| \leq \| A \| \| \Delta x \| \| \Delta b \| \)

\[
\| \Delta x \| \leq \| A \| \| \Delta b \|
\]

So this tells us about metric properties.