Unitary operators: invertible isometries \( U^* U = I \).

Use generalized eigenvalue identity: if \( U: X \to Y \) takes an orthonormal basis \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) to \( U\vec{v}_1, U\vec{v}_2, \ldots, U\vec{v}_n \), an orthonormal basis.

**Proposition:** Let \( U \) be a unitary matrix.

1. \[ |\det U| = 1 \]
2. If \( \lambda \) is an eigenvalue of \( U \) then \( |\lambda| = 1 \).

**Proof:**

1. \[ 1 = \det (U^* U) = \det(U^*) \det(U) = \det(U^t) \det(U) = \det(U)^2 \]

2. If \( U\vec{v} = \lambda \vec{v} \)

\[ ||U\vec{v}|| = ||\lambda \vec{v}|| = |\lambda| ||\vec{v}|| \]

\[ ||\vec{v}|| = \sqrt{a^2 + b^2} \]

So \( |\lambda| = 1 \). Since \( U^* \) is the inverse of \( U \), and \( \det(U^*) = \det(U) \), we have \( |\det(U)| = 1 \).

Operators \( A \) and \( B \) are called unitarily equivalent if \( A = UBV^* \) for some unitary \( U \). (Note \( U^* = U^t \))

So unitarily equivalent operators are similar.

**Proposition:** A matrix \( A \) is unitarily equivalent to a diagonal matrix \( D \) if \( A \) has an orthonormal (orthonormal) basis of eigenvectors.

**Proof:** First note that if \( A = UBV^* \) and \( B\vec{x} = \lambda \vec{x} \) for \( \lambda \neq 0 \), then \( AU\vec{x} = (UBV^*)U\vec{x} = UBV^*U\vec{x} = UBV^*\lambda \vec{x} = U\lambda \vec{x} \).

So \( U\lambda \vec{x} \) is an eigenvector of \( A \).

If \( A = UDU^* \) where \( D \) is diagonal, \( U \) unitary, since \( \vec{e}_1, \ldots, \vec{e}_n \) are a standard orthonormal basis of eigenvectors for \( D \),

\[ D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \]

\[ U\vec{e}_k = e_k \]

So \( U\vec{e}_1, U\vec{e}_2, \ldots, U\vec{e}_n \) is an orthonormal basis of eigenvectors for \( A \).
Theorem: Let $A : X \rightarrow X$ be an operator acting on a complex inner product space. Then there exists an orthonormal basis $\tilde{v}_1, \ldots, \tilde{v}_n$ such that $A$ is upper triangular with respect to this basis. That is, $A = U^*TV^*$ where $U$ is unitary, upper triangular.

Proof: by induction on the dimension of $X$.

So if $\dim X = 1$, then $A$ is $1 \times 1$, thus upper triangular.

Suppose it's true for $\dim X = n-1$. Let $\tilde{v}_1$ be an eigenvector of $A$ with eigenvalue $\lambda_1$, such that $\|\tilde{v}_1\| = 1$.

Let $E = (\text{span} \{ \tilde{v}_1 \})^\perp$, that is all vectors orthogonal to $\tilde{v}_1$.

Since $X = E + E^\perp$, $\dim E = n-1$ and $E$ has an orthonormal basis $\tilde{v}_2, \tilde{v}_3, \ldots, \tilde{v}_n$.

In this basis, $A_{ij} = \frac{\lambda_1}{\tilde{v}_i, \tilde{v}_j}$ (if $A_{ij}$ is upper triangular).

Then so is $A$. Thus, $A_{ij} = \frac{\lambda_1}{\tilde{v}_i, \tilde{v}_j}$. Therefore, $A = U^*TV^*$. 

Thus, $A$ is upper triangular.
notice $A: E \to E$.
by identifying $E$ with vectors whose first coordinate is 0.

$\begin{bmatrix}
0 \\
\ddots \\
0
\end{bmatrix}$

Since $\dim E = n - 1$
exists an orthonormal basis $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ such that $A_1$ is upper triangular.

so $A$ is upper triangular with respect to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

-note: $\mathbf{u}$ and $T$ may have complex entries even if $A = UTU^*$ is real.
-note: I didn't need to make the basis vectors orthonormal (it was just cool.)

**Theorem (keepin' it real)**

Let $A: x \to x$ be an operator on a real inner product space.
Suppose all eigenvalues of $A$ are real,
then exists orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ such that $A$ is upper triangular and real

$A = UTU^*$

**Proof:** same as last theorem except when I get to

$\begin{bmatrix}
\lambda I \\
0
\end{bmatrix}$
to use the inductive step show $A_1$ has real eigenvalues.

$\det (A - \lambda I) = (\lambda - \lambda_1) \det (A_1 - \lambda I)$
so eigenvalues of $A_1$ are eigenvalues of $A$, thus real,
so we can use inductive step.