Proof of Lemma: Flat rotation $\Rightarrow$ let $\sum A_k = \bar{v}^i_1, \bar{v}^i_2, ..., \bar{v}^i_p = A$ and split $\sum i \bar{v}^i_j$ into subsets $\sum_{A_k}$ s.t. $A_k = \sum_{j \in \Lambda_k} \bar{v}^i_j$, $j \in \Lambda_k$.

Skip to Criterion of Diagonalizability

If matrix $A^T$ is diagonalizable then the characteristic polynomial has $n$ roots: $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) ... (\lambda_n - \lambda)$ in fact, over $\mathbb{C}$, this is true over any matrix.

**Theorem 2.8:** An operator $A: V \rightarrow V$ is diagonalizable iff every eigenvalue $\lambda$ has an algebraic multiplicity equal to the dimension of its eigen space, $\dim \ker (A - \lambda I)$. (geometric multiplicity)

Rest of proof in the back.

3/19/80

Criterion of Diagonalizability

We know (over $\mathbb{C}$) any characteristic polynomial of $A: V \rightarrow V$ will have $n = \dim V$ eigenvalues (counting multiplicities)

$\det (A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2) ... (\lambda - \lambda_n)$

diagonalizable

**Criterion Theorem:** An operator $A: V \rightarrow V$ is diagonalizable iff

$\forall \lambda \in \sigma(A)$, $\dim \ker (A - \lambda I) =$ algebraic multiplicity of $\lambda$

Proof: If $A$ is diagonalizable then in some basis, it looks like

$\begin{pmatrix}
\lambda_1 & 0 & & \\
0 & \lambda_2 & & \\
& & \ddots & \\
0 & & & \lambda_n
\end{pmatrix}$

so for every occurrence of $\lambda$ in characteristic polynomial, we have a linearly independent eigenvector

Proved in one direction.

Other way: Let $\lambda_1, ..., \lambda_r$ be eigenvalues of $A$ and let $E_1, ..., E_r$ be eigenspaces ($E_k = \ker (A - \lambda_k I)$) and suppose $\dim E_k =$ algebraic multiplicity of $\lambda_k$.

Then by "another theorem," the $E_k$s are linearly independent, and $\dim \bigoplus E_k = \sum$ algebraic multiplicities.

So, $\bigoplus E_k$ be a basis for $V$. 

$\rightarrow$
Then \( \bigcup_{k=1}^{n} F_k \) is linearly independent in \( V \) by the theorem, and these \( x_k \) are \( n \) vectors in \( \bigcup_{k=1}^{n} F_k \), thus a basis for \( V \).

So, \( A \) is D-able.
CHAPTER 5 - INNER PRODUCT SPACES

5.1

We define the length of a vector \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) by the Pythagorean rule:

\[
\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}
\]

for \( x \in \mathbb{R}^n \),

\[
\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
\]

The dot product in \( \mathbb{R}^3 \) is defined as:

\[
x \cdot y = x_1y_1 + x_2y_2 + x_3y_3
\]

Leslie:

In \( \mathbb{R}^n \), the inner product 

\[
(x, y) = x_1y_1 + \cdots + x_ny_n
\]

\[
-(x, x) = \|x\|^2
\]

for \( \mathbb{C}^n \), for a complex \( z = a + ib \in \mathbb{C} \)

\[
|z|^2 = a^2 + b^2 \rightarrow \text{complex conjugate}
\]

So, if

\[
z = \left( \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_n \end{array} \right) \in \mathbb{C}^n
\]

\[
z = \left( \begin{array}{c} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{array} \right)
\]

Thus,

\[
\|z\|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2
\]

So, norm on \( \mathbb{C}^n \) is like norm on \( \mathbb{R}^{2n} \)

for an inner product on \( \mathbb{C}^n \), we want

\[
(z, z) = \|z\|^2
\]

So we choose for \( \bar{z}, \bar{w} \in \mathbb{C} \),

\[
(z, \bar{w}) = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n
\]

So,

\[
(z, \bar{w}) = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n
\]

We define the Hermitian Adjoint (Adjoint*): \( A^* \) of matrix \( A \) as 

\[
A^* = \overline{A^T}
\]

So, for \( \theta \in \mathbb{C}, \bar{z}, \bar{w} \in \mathbb{C}^n \), we have

\[
(z, \bar{w}) = \bar{w}^* z^* = (\bar{z}, \bar{w}) = \overline{\bar{z}^* \bar{w}} = (\bar{\bar{w}}, \bar{z}) \rightarrow \text{conjugated}
\]
Inner Product Spaces

In $\mathbb{R}^n$ and $\mathbb{C}^n$, the inner product satisfies:
1) Conjugate symmetry: $(\alpha, \beta) = (\overline{\beta}, \alpha)$
2) Linearity in the first vector: $(\alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z}) = \alpha (\mathbf{x}, \mathbf{z}) + \beta (\mathbf{y}, \mathbf{z})$
3) Non-negativity: $(\mathbf{x}, \mathbf{x}) \geq 0$ for all $\mathbf{x}$ (also real)
4) Non-degeneracy: $(\mathbf{x}, \mathbf{x}) = 0$ iff $\mathbf{x} = \mathbf{0}$

Let $V$ be an inner product space. Then there is an inner product on $V$ that is a function of two vectors with the above properties.

From the inner product we define the norm $\|\mathbf{x}\|^2 = (\mathbf{x}, \mathbf{x})$

Exs 1) standard inner product on $\mathbb{R}^n$ and $\mathbb{C}^n$

$$ (\mathbf{x}, \mathbf{y}) = \mathbf{x}^* \mathbf{y} $$

2) Let $V = \mathbb{P}_n$ for every $f, g \in \mathbb{P}_n$

$$ (f, g) = \int_0^1 f(t) g(t) \, dt $$

3) For $V = M_{m \times n}$ and $A, B \in V$

$$ (A, B) = \text{trace}(B^* A) $$

More Properties

If $(\mathbf{x}, \alpha \mathbf{y} + \beta \mathbf{z}) = (\alpha \mathbf{y} + \beta \mathbf{z}, \mathbf{x}) = \alpha (\mathbf{x}, \mathbf{y}) + \beta (\mathbf{x}, \mathbf{z})$

Lemma 4.1: Let $\mathbf{x} \in V$, $V$ is an inner product space.

Then, $\mathbf{x} = \mathbf{0}$ iff $(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{y} \in V$

Proof: If $\mathbf{x} = \mathbf{0}$, then $(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{y}$

Other way: Suppose $(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{y} \in V$

Then $\mathbf{x} = \mathbf{0}$.

Let $\mathbf{y} = \mathbf{x}$, then $(\mathbf{y}, \mathbf{y}) = \|\mathbf{y}\|^2 = 0 \implies \mathbf{x} = \mathbf{0}$

Corollary: $\mathbf{x} = \mathbf{y}$ iff $(\mathbf{x}, \mathbf{z}) = (\mathbf{y}, \mathbf{z})$ for all $\mathbf{z} \in V$

Proof: Use prev. lemma but with $\mathbf{x} = \mathbf{y}$ instead of $\mathbf{x} = \mathbf{0}$

If $\mathbf{x} - \mathbf{y} = \mathbf{0}$, we have $(\mathbf{x} - \mathbf{y}, \mathbf{z}) = 0$ for all $\mathbf{z} \in V$