Suppose an operator (matrix) $A: V \to V$ has a basis $\{v_1, \ldots, v_n\}$ of eigenvectors with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

Then $[A]_{\beta} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Thus $[A]_{\beta} = \begin{pmatrix} [A_{v_1}]_{\beta} & \cdots & [A_{v_n}]_{\beta} \end{pmatrix} = \begin{pmatrix} [A_{v_1}]_{\beta} & \cdots & [A_{v_n}]_{\beta} \end{pmatrix}$.

$[\vec{v}_k]_{\beta} = e_k$ maps to standard basis.

So $[A]_{\beta} = \begin{pmatrix} \lambda_1 e_1 & \cdots & \lambda_n e_n \end{pmatrix}$.

$[A^N]_{\beta} = \begin{pmatrix} \lambda_1^N e_1 & \cdots & \lambda_n^N e_n \end{pmatrix} = \text{diag}(\lambda_1^N, \lambda_2^N, \ldots, \lambda_n^N)$.

$e^{t A} = I + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + \ldots$ power series expansion of $e^x$.

$[e^{tA}]_{\beta} = \begin{pmatrix} e^{t \lambda_1} e_1 & \cdots & e^{t \lambda_n} e_n \end{pmatrix} = \begin{pmatrix} e^{t \lambda_1} e_1 & \cdots & e^{t \lambda_n} e_n \end{pmatrix}$.
Let \( S \cdot [I]_{
\text{sp}} \).

\[
A = [A]_{\text{sp}} = [I]_{\text{sp}} [A]_{\text{pp}} [I]_{\text{sp}}^{-1}
\]

D: the drag matrix

\( A = SD^{-1} S^{-1} \)

\( A^n = SD^n S^{-1} \)

\( A^n = (SD^n S^{-1})(SD^n S^{-1}) \ldots (SD^n S^{-1}) \) (n times)

\( = SD^n S^{-1} \)

\( e^{\lambda A} = S e^{\lambda D} \)

Theorem: A matrix \( A \) admits a representation \( A = SD^n S^{-1} \) iff there is a basis of eigenvectors of \( A \).

In this case, we call \( A \) diagonalizable.

We already showed that if \( A \) has a basis of eigenvectors, a representation exists. Now we need to show if \( A = SD^n S^{-1} \) for a diagonal matrix \( D \)

Then \( A \) admits a basis of eigenvectors of \( A \).
Suppose $A = SDS^*$

then $AS = SD$  $D$ is a diag matrix

$S = [s_1, s_2, ..., s_n]$

then $AS = SD$

$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$

$[A s_1, A s_2, ..., A s_n]$

$[\lambda_1 s_1, \lambda_2 s_2, ..., \lambda_n s_n]$

So we see $s_1, ..., s_n$ are eigenvectors and a basis.

Because $S$ is invertible, $S$ is a basis of the standard.

Also

Theorem: Let $\lambda_1, ..., \lambda_n$ be distinct eigenvalues with eigenvalues $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ resp.

Then the vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ are lin indep.
Induction

To prove a statement $P(n)$ for all $n \in \mathbb{N}$, use induction:

1. **Base Case:** Show $P(1)$ is true for the smallest case.

2. **Inductive Step:** Suppose $P(k)$ is true for arbitrary $k$. Show $P(k+1)$ follows.

**Proof of Theorem:** We will use induction on $n$. Let $P(n)$ be the theorem.

$P(1)$ is true. If we have an eigenvector $\mathbf{v}$ with eigenvalue $\lambda$, then $\mathbf{v}$ is nonzero, thus by inductive hypothesis...
Suppose $P(k-1)$ is true. Show $P(k)$.

Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues with eigenvectors $v_1, \ldots, v_k$. We want to show $\lambda$ is in $\text{lin}$ and

Suppose $\sum_{j=1}^{k} c_j v_j = 0$ for scalars

$$\left( A - \lambda_k I \right) \left( \sum_{j=1}^{k} c_j v_j \right) = \sum_{j=1}^{k} c_j (\lambda_j - \lambda_k) v_j$$

$$= \sum_{j=1}^{k-1} c_j (\lambda_j - \lambda_k) v_j = 0$$

We know this isn't 0 since they're distinct.

Since $P(k-1)$ is true $v_1, \ldots, v_{k-1}$ are lin indep.

Thus $c_j (\lambda_j - \lambda_k) = 0$ for all $1 \leq j \leq k-1$.

So $c_j = 0$ for all $1 \leq j \leq k-1$.

And $v_k = 0$ so for $\sum_{j=1}^{k} c_j v_j = 0$, $c_k = 0$ too.

Thus this is the linear lin combination.