Chapter 4 - "Introduction to Spectral Theory" (Eigenvalues and Eigenvectors)

"The main idea of spectral theory is to split an operator (matrix) into [manageable chunks]."

Consider the difference equation $A: V \rightarrow V$ \( x_{n+1} = A x_n \); (i.e. \( x_1 = A x_0, x_2 = A x_1, \ldots \))
Note that \( x_n = A^n x_0 \); we want to observe the behavior of this as \( n \) gets large without needing to compute \( A^n \) for large \( n \).
What if we were to find a vector \( \mathbf{v} \) such that \( A \mathbf{v} = \lambda \mathbf{v} \) for some scalar \( \lambda \)? Therefore, \( A \mathbf{v} = \lambda \mathbf{v} \).

A scalar \( \lambda \) is called an eigenvalue of an operator \( A: V \rightarrow V \) if \( \exists \) a nonzero vector \( \mathbf{v} \in V \) s.t. \( A \mathbf{v} = \lambda \mathbf{v} \); \( \mathbf{v} \) is called an eigenvector of \( A \), corresponding to \( \lambda \).
Once we find \( \lambda \), we solve \( A \mathbf{x} = \lambda \mathbf{x} \) to find the eigenvectors.

If \( A \mathbf{x} = \lambda \mathbf{x} \) for nonzero \( \mathbf{x} \in V \), \( A \mathbf{x} = (A \mathbf{I}) \mathbf{x} \) and \( (A - \lambda \mathbf{I}) \mathbf{x} = \mathbf{0} \) for nonzero \( \mathbf{x} \in V \).
This implies \( \det (A - \lambda \mathbf{I}) = 0 \), which is a polynomial in \( \lambda \); \((A - \lambda \mathbf{I})\) has a nontrivial kernel, so it's not invertible.

Let \( \ker (A - \lambda \mathbf{I}) \) be called the eigenspace of \( \lambda \). The set of eigenvalues for \( A \) is called the spectrum of \( A \), and is denoted \( \sigma (A) \).

To find eigenvalues, we want to find zeroes of \( P(\lambda) = \det (A - \lambda \mathbf{I}) \).
In short, \( \lambda \in \sigma (A) \iff \det (A - \lambda \mathbf{I}) = 0 \).

If \( A \) is an \( n \times n \) matrix, it can be shown that \( P(\lambda) \) is a polynomial of degree \( n \), called the characteristic polynomial of \( A \).

Now, suppose we have a more abstract linear transformation \( T: V \rightarrow V \) in an abstract vector space. How do we compute the spectrum? Pick a basis, get a matrix \( [T]_{BB} \), and then find \( \lambda \) using \( \det ([T]_{BB} - \lambda \mathbf{I}) \).

\[ \text{Remember the change of basis: } [T]_{BB} = [T]_{BA} [T]_{AA} [T]_{AB} = [T]_{BA} [T]_{AA} [I]_{BA}^{-1}, \text{ this takes us from basis } A \text{ to basis } B. \]\nNote that \([T]_{BB} \text{ and } [T]_{AA} \text{ are similar, as there exists an invertible matrix } S \text{ such that } [T]_{BB} = S [T]_{AA} S^{-1}. \]

Suppose \( A \) and \( B \) are similar. Then \( A = S B S^{-1} \) for some invertible \( S \).
\( A - \lambda \mathbf{I} = S B S^{-1} - \lambda \mathbf{I} = S (B - \lambda \mathbf{I}) S^{-1}, \) so \( A - \lambda \mathbf{I} \) is similar to \( B - \lambda \mathbf{I}, \) so \( \det (A - \lambda \mathbf{I}) = \det (B - \lambda \mathbf{I}) \) and \( A \) and \( B \) have the same characteristic polynomial and spectrum.
This implies that the spectrum is independent of basis choice.

Let \( P(x) = \det(A-xI) \). If \( \lambda \in \sigma(A) \), then \( P(\lambda) = 0 \), as \( \lambda \) is an eigenvalue of \( A \).

\[ P(x) = (x-\lambda)^k q(x) \]

where \( q(x) \) is a polynomial that doesn't have \( \lambda \) as a root.

\( k \) is the algebraic multiplicity (often called "multiplicity") of \( \lambda \).

Note that \( P(x) = \sum_{k=0}^{\infty} a_k x^k \) as an \( n \)-degree polynomial has exactly \( n \) roots in \( \mathbb{C} \), "counting multiplicities".

That is, \( P(x) = a_n(x-\lambda_1)(x-\lambda_2)...(x-\lambda_n) \); there are \( n \) of \( (x-\lambda_i) \), some might be repeated.

Also, \( \dim \text{Ker}(A-\lambda I) \) is called the geometric multiplicity.

Proposition: geometric multiplicity of \( \lambda \) \( \leq \) algebraic multiplicity of \( \lambda \)

Proof: Homework problem 1.9

Proposition: If \( A \) is an \( n \times n \) matrix, and \( \lambda_1, \ldots, \lambda_n \) are its eigenvalues (counting multiplicities), then:

1) \( \text{Trace } A = \lambda_1 + \lambda_2 + \cdots + \lambda_n \)
2) \( \det A = \lambda_1 \lambda_2 \cdots \lambda_n \)

Proof: Homework problems 1.10 and 1.11

Eigenvalues of a triangular matrix - If \( A \) is triangular with diagonal entries \( a_1, \ldots, a_n \), then \( (A-\lambda I) \) is triangular with diagonal entries \( (a_1-\lambda), (a_2-\lambda), \ldots, (a_n-\lambda) \), and \( \det (A-\lambda I) = (a_1-\lambda)(a_2-\lambda)\cdots(a_n-\lambda) \), so the eigenvalues are the diagonal entries.