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Basic Properties of Determinant

1. The determinant is linear in each column vector (multilinear)
   \[ D(v_1, \ldots, \alpha v_k + \beta v_{k+1}, \ldots, v_n) = \alpha D(v_1, \ldots, v_k, \ldots, v_n) + \beta D(v_1, \ldots, v_{k+1}, \ldots, v_n) \]

2. The determinant is anti-symmetric, if we switch columns (swap, transpose, interchange) then
   \[ D(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -D(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n) \]

3. Normalization. \( D(e_1, e_2, \ldots, e_n) = 1 \) \( \forall e_i \in \mathbb{R}^n \)

Now to derive the other properties.

Proposition: Let \( A \) be a square matrix

1. If \( A \) has a zero column, \( \det A = 0 \)
2. If \( A \) has two equal columns, \( \det A = 0 \)
3. If one column of \( A \) is a multiple of another, \( \det A = 0 \)
4. If columns of \( A \) are linearly dependent (if \( A \) is not invertible), then \( \det A = 0 \)

Proof of 1: Since \( (e) \hat{e} = \hat{e} \)
   \[ D(v_1, \ldots, e, \ldots, v_n) = 0 \cdot D(v_1, \ldots, e, \ldots, v_n) = 0 \]

Proof of 2: If \( A \) has two columns that are the same, then
   \[ \det A = -\det A \] so \( \det A = 0 \)

Proof of 3: \[ D(v_1, \ldots, v_k, \ldots, v_{k+1}, \ldots, v_n) = \alpha D(v_1, \ldots, v_k, \ldots, v_{k+1}, \ldots, v_n) = 0 \]

Proof of 4: Let \( \hat{v}_j \in \mathbb{R}^n \) then
   \[ D(v_1, \ldots, \alpha v_j + \beta v_k, \ldots, v_n) = \sum_{j=1}^n \hat{v}_j D(v_1, \ldots, v_j, \ldots, v_n) = 0 \]
   Each one is zero because of a repeating vector.
We have column replacement because if \( \mathbf{u} = \sum_{j=1}^{k} a_j \mathbf{v}_j \)
then \( D(\mathbf{u}, \ldots, \mathbf{v}_k+\mathbf{u}, \ldots, \mathbf{v}_n) = D(\mathbf{u}, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_n) \cdot D(\mathbf{v}_k+\mathbf{u}, \ldots, \mathbf{v}_n) \).

\[ \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn} \]

diagonal matrix

\[ = D(a_{11}, a_{22}, \ldots, a_{nn}) = a_{11}a_{22}\cdots a_{nn} \]

The determinant of a diagonal matrix is the product of the diagonal entries.

An **upper triangular** matrix is a matrix with only zero entries below the diagonal.

\[ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \]

and

\[ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \]

lower triangle

A **triangular matrix** is a matrix with only zero entries either above or below the diagonal.

The determinant of a triangular matrix is the product of the diagonal entries.

*Proof*: If one of the diagonal entries is zero then that column is linearly dependent on the others.

Otherwise, column replacement can turn a triangular matrix into a diagonal matrix without changing the diagonal.

We can switch columns \( \Rightarrow \) multiplying determinant by -1

We can scale columns \( \Rightarrow \) scaling determinant

We can do column replacement \( \Rightarrow \) does not change determinant

**Strategy**: Get triangular matrix by column replacement

Keeping track of scalings and swaps.
If $A$ is invertible then every column of $(A^T)e$ has a pivot, and since $A$ is square that means $(A^T)e$ has a nonzero determinant.

Proposition: $\det A \neq 0$ iff $A$ is invertible $\implies \det A \neq 0$ iff $A$ is not invertible.

Theorem: $\det (A) = \det (A^T)$

Theorem: $\det (AB) = \det (A) \det (B)$

To prove this we need lemmas.

Lemma: For a square matrix $S$ and an elementary matrix of the same size,

$$\det (SE) = (\det A)(\det E)$$

Elementary matrices when acting on the left, $EA$ corresponds to row operations and you can check that $AE$ corresponds to column operations.

(see Section 2.2 for descriptions of $E$)

Lemma: For an elementary matrix $E$, $\det (E) \cdot \det (E^T)$ is true because $E$ is triangular or symmetric.

Now we prove the theorems with Theorem 4.1 from Section 2.4.

Any invertible matrix is the product of elementary matrices.