Rank theorem: For a matrix \( A \), \( \text{rank } A = \text{rank } A^T \).

Proof: \( \text{rank } A = \# \text{ of pivots in } A \). Similarly, \( \text{rank } A^T = \# \text{ of pivots in } A^T \).

Rank-nullity theorem: If \( A \) is an \( m \times n \) matrix, \( A: \mathbb{R}^n \rightarrow \mathbb{R}^m \), then:

1) \( \text{dim } \text{ker } A + \text{dim } \text{ran } A = \text{dim } \text{ker } A^T + \text{rank } A^T = n \)
2) \( \text{dim } \text{ker } A + \text{dim } \text{ran } A^T = \text{dim } \text{ker } A^T + \text{rank } A = m \)

Proof: Since \( \text{rank } A = \# \text{ of pivots in } A \) and \( \text{dim } \text{ker } A^T = \# \text{ of free variables in } A^T \),
then \( \text{rank } A + \text{dim } \text{ker } A^T = \# \text{ of columns in } A = n \).

For part (b), use \( B^T: \mathbb{R}^m \rightarrow \mathbb{R}^n \), in part (a).

From this theorem, we see that \( \text{ker } A \) "kills" dimensions from \( \mathbb{R}^n \) to \( \text{ran } A \).

Check out the example that follows in the book.

Theorem: If \( A \) is an \( m \times n \) matrix, \( A: \mathbb{R}^n \rightarrow \mathbb{R}^m \), then \( AB = 0 \) has a solution \( \forall \alpha \in \mathbb{R}^m \)

If \( AT = 0 \) has only the trivial solution. \( \square \)

How to complete a basis: If you have a set of linearly independent vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \), let \( A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \) be the matrix made of the row vectors. If \( \text{ran } A \)
in echelon form is \( [\text{free variables set to free variables}] \\
\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \alpha_1 \\
0 & \cdots & 0 & \alpha_2 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \alpha_k \end{bmatrix} \)

Suppose we have \( A\vec{x} = \begin{bmatrix} 1 & 2 & 3 & 1 \\
0 & 0 & 1 & 6 \end{bmatrix} \).

Completed basis: \( \begin{bmatrix} 1 & 2 & 3 & 1 \\
0 & 0 & 0 & 6 \end{bmatrix} \)

2.8

Until now (mostly) our column vectors have referred to objects in \( \mathbb{R}^n \).
\[
[\begin{bmatrix} x_1 \\
x_2 \\
x_3 \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \frac{2}{3} \times \vec{x} \in \mathbb{R}^3. \text{ If } V \text{ has a basis } \vec{b}_1, \vec{b}_2, \vec{b}_3, \text{ then } \vec{c} = \begin{bmatrix} 3 \\
2 \\
1 \end{bmatrix} \text{ can be written as } \begin{bmatrix} 3 \\
2 \\
1 \end{bmatrix} = \begin{bmatrix} 3 \vec{b}_1 + 2 \vec{b}_2 + \vec{b}_3 \end{bmatrix}.
\]

The map \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) is an isomorphism from \( V \) to \( W \) if \( \vec{x} \in \mathbb{R}^n \).

Let \( T: V \rightarrow W \) be a linear transformation, let \( A = [\vec{b}_1, \vec{b}_2, \vec{b}_3] \) be a basis for \( V \) and \( B = [\vec{c}_1, \vec{c}_2, \vec{c}_3] \) be a basis for \( W \). Then \( \begin{bmatrix} \vec{b}_1 \\
\vec{b}_2 \\
\vec{b}_3 \end{bmatrix} = \begin{bmatrix} \vec{c}_1 \\
\vec{c}_2 \\
\vec{c}_3 \end{bmatrix} \). We see that \( [T\vec{b}_1]_B = [T\vec{c}_1]_B [T\vec{b}_1]_B = [T\vec{c}_2]_B [T\vec{b}_2]_B = [T\vec{c}_3]_B [T\vec{b}_3]_B \).

Composition of linear transformations is still matrix multiplication:
\[
T_1: X \rightarrow Y, \quad T_2: Y \rightarrow Z, \quad T_3: X \rightarrow Z.
\]

And \( A, B, C \) are bases for \( X, Y, Z \) respectively, then \( [T_2 \circ T_1]_A = [T_2]_B [T_1]_A \). (notice the index cancellation). The point of this is the same as in \( \mathbb{R}^n \). \( \square \)
Let a vector space V have two bases $A = \{e_1, \ldots, e_n\}$, $B = \{f_1, \ldots, f_n\}$, then the identity map $I: V \rightarrow V$ can be written in terms of these bases, $[I]_A^B [I]_B^A = I$. This is called the change of coordinate matrix. We see $[I]_B^A = [I]_A^B [B]_B^A$. What is $[I]_B^A$? It is $[I]_A^B$.

If $T: V \rightarrow W$ is a linear trans. and $V$ has basis $A$, $W$ has basis $B$, then $[T]_B^A = [I]_B^A [T]_A^B [I]_B^A$.

If $T: V \rightarrow V$ and $V$ has a basis $A$ then we say $[T]_A$ for $[T]_A^A$. If $B$ is another basis for $V$, then $[T]_B^A = [I]_B^A [T]_A^B [I]_B^A$, and $[T]_B^B = Q [T]_A^B Q^{-1}$, where $Q = [I]_A^B$. We say two matrices $A$ and $B$ are similar if there exists an invertible matrix $Q$ such that $B = Q^{-1} A Q$. 