Section 2.3: Analyzing Pivots (continued)

Given a matrix $A$, let $A_e$ be an echelon form of $A$. Let $A = [\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m]$ for each $\tilde{v}_k \in \mathbb{R}^m$.

We showed
1. $\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m$ are **linearly independent**
   **iff** there is a pivot in every column of $A_e$,
2. $\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m$ are **spanning**
   **iff** there is a point in every row of $A_e$,
3. $\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m$ is a **basis**
   **iff** $\exists$ a pivot in every row and column of $A_e$.

**PROP:** Any linearly independent system of vectors in $\mathbb{R}^n$ has no more than $n$ vectors in it.

**PROOF:** Let $\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m$ be a linearly independent in $\mathbb{R}^m$. Let $A = \begin{bmatrix} | & | & | \\ \tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m \end{bmatrix}$. There is a pivot point in every column, but there are only $n$ rows. There is a pivot in $m$ rows, so $m \leq n$. ■

**PROP:** Any two bases in $V$ have the same number of vectors; we will call this number the **dimension** of $V$, or **dim V**.

**PROOF:** Let $\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_n$ and $\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_m$ be bases for $V$. WLOG, let $n \leq m$. Let $A: \mathbb{R}^n \to V$ where $A\tilde{v}_k = \tilde{v}_k$ for all $k$. $A$ is an isomorphism (sends basis to basis). So $A^{-1}\tilde{w}_1, A^{-1}\tilde{w}_2, ..., A^{-1}\tilde{w}_m$ is a basis for $\mathbb{R}^n$ because $A^{-1}$ is an isomorphism. So $m \leq n$. So $m = n$. ■

**PROP:** Any basis in $\mathbb{R}^n$ has $n$ vectors in it.

**PROOF:** Use previous proposition.

**PROP:** Any spanning set in $\mathbb{R}^n$ must have at least $n$ vectors in it.

**PROOF:** Let $\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m$ be a spanning set in $\mathbb{R}^n$ and let $A = \begin{bmatrix} | & | & | \\ \tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m \end{bmatrix}$ be an $n \times m$ matrix. So $A_e$ has a pivot in every row ($n$ rows) so it has...
a pivot in \( n \) columns. So \( n \leq m = \text{total number of columns} \).

**PROP:** A matrix \( A \) is invertible iff \( A \) has a pivot in every column and every row.

**PROOF:** On Friday, this claim about pivots is equivalent to \( \forall \, \vec{b} \in \mathbb{W} \exists \, \vec{x} \in \mathbb{V} \) such that \( A\vec{x} = \vec{b} \). This means \( A \) is invertible.

**CORR:** An invertible matrix must be square (that is, it must be \( n \times n \)).

**PROP:** If \( A \) is square and left-invertible or right-invertible, then it is invertible.

**PROOF:** As an exercise.

**Section 2.4 : Finding the Inverse of \( A \)**

Suppose \( A \) is an invertible \( n \times n \) matrix. Then we write the \( n \times 2n \) augmented matrix \( ( A \mid I_n ) \). Then use row reduction to turn \( A \) into \( I_n \). You will get \( ( I_n \mid A^{-1} ) \).

Why does this work?

First reason. Since \( I_n = \begin{bmatrix} | & | & | \\ \vec{e}_1, \vec{e}_2, ..., \vec{e}_n \end{bmatrix} \) by row reducing \( ( A \mid I_n ) \) we are solving \( A\vec{x} = \vec{e}_k \) \( \forall k \) simultaneously. So \( \vec{x} = A^{-1}\vec{e}_k \), so the output is \( ( I_n \mid [A^{-1}\vec{e}_1, A^{-1}\vec{e}_2, ..., A^{-1}\vec{e}_n] ) = ( I_n \mid A^{-1} ) \).

Second reason. Let \( E_1, E_2, ..., E_N \) be the elementary matrices (i.e. row operations) that turn \( A \) into \( I_n \). Let \( E = E_N \ldots E_2 E_1 \), so \( EA = I_n \), so \( A = E^{-1} = E_1^{-1}E_2^{-1} \ldots E_N^{-1} \) so \( A^{-1} = E \).

**PROP:** Every invertible matrix is the product of elementary (row operation) matrices.

**PROOF:** See above.
(Also see book for example.)

Section 2.5: Dimensions
The number of basis vectors in $V$ is the dimension of $V$, called $\text{dim } V$. For example, $\text{dim}\{0\} = 0$. If $V$ does not have a finite basis, then $\text{dim } V = \infty$, $V$ is infinite-dimensional (otherwise, it is finite dimensional.)

\textbf{PROP:} A vector space is finite dimensional \textbf{iff} it has a finite spanning system.

\textbf{PROOF:} Basis is spanning and you can find a basis inside a spanning system.

\textbf{PROP:} If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ is linearly independent in $V$ and $\text{dim } V = n$, we can find $\vec{v}_{r+1}, \vec{v}_{r+2}, \ldots, \vec{v}_n$ such that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{r+1}, \vec{v}_{r+2}, \ldots, \vec{v}_n$ is a basis.