Proposition: a system $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in V$

is a linearly dependent if and only if

b) one of the vectors is a linear combination of the others
$$\vec{v}_k = \sum_{j \neq k}^p \beta_j \vec{v}_j$$

Proof

a) $\implies$ b) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are linearly dependent then $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots + \alpha_p \vec{v}_p = \vec{0}$ where at least one $\alpha_k \neq 0$

Pick $\alpha_k$ to be the "at least one" scalar and subtract $\alpha_k \vec{v}_k$ from both sides
$$\sum_{j \neq k}^p \alpha_j \vec{v}_j = -\alpha_k \vec{v}_k \implies -\sum_{j \neq k}^p \frac{\alpha_j}{\alpha_k} \vec{v}_j = \vec{v}_k$$
call $\beta_j = -\frac{\alpha_j}{\alpha_k}$

b) $\implies$ a) Suppose $\sum_{j \neq k}^p \beta_j \vec{v}_j = \vec{v}_k$ for some $k.$

$$\sum_{j \neq k}^p \beta_j \vec{v}_j - \vec{v}_k = \vec{0}$$

w/ nonzero coefficient of $-1$

Thus, the set $\vec{v}_1, \ldots, \vec{v}_p$ is linearly dependent

Proposition: A system $\vec{v}_1, \ldots, \vec{v}_n \in V$

(use in hw 2.5) is a basis if

b) linearly independent and generating

Proof

a) $\implies$ b) For a basis, every vector $\vec{v} \in V$ has a unique representation
$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots + \alpha_n \vec{v}_n = \vec{v}$$

since $\vec{v} \in V$ is any vector, our basis is a spanning set.

If we choose $\vec{v} = \vec{0}$ then we have the unique, trivial representation $0_1 \vec{v}_1 + 0_2 \vec{v}_2 + \ldots + 0_n \vec{v}_n = \vec{0}$

This set is linearly independent.
b) \[ \text{Let } \mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_n \text{ be linearly independent and generating.} \]

pick any \( \mathbf{v} \in \mathbf{V} \)

since the set spans \( \mathbf{V} \), we have a representation
\[ a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \ldots + a_n \mathbf{v}_n = \mathbf{v} \]

suppose we had another representation \( b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \ldots + b_n \mathbf{v}_n = \mathbf{v} \)

subtract one from the other:
\[ \mathbf{v}_1 - \mathbf{v}_1 = 0 = (a_1 - b_1) \mathbf{v}_1 + (a_2 - b_2) \mathbf{v}_2 + \ldots + (a_n - b_n) \mathbf{v}_n \]

\( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) is linearly independent, so this is the trivial solution, meaning that \( a_k - b_k = 0 \) for all \( k \)
\[ a_k = b_k \]
so the representations are the same.

Proposition: Any finite generating set \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) contains a basis as a subset.

Proof

Suppose \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) is linearly independent then it's a basis... we're done.

Suppose \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) is linearly dependent then there exists \( \mathbf{v}_k \) in our set such that \( \mathbf{v}_k = \sum_{i=1}^{p} \beta_i \mathbf{v}_i \)

Throw \( \mathbf{v}_k \) out.

Repeat the process until you have a linearly independent generating set. This is a basis!

Section 3: Linear Transformations

A transformation \( (T) \) aka map, operator, function takes input from one set \( \mathbf{X} \) and gives output in another
\[ T(\mathbf{x}) = \mathbf{y} \in \mathbf{Y} \]

\( T: \mathbf{X} \to \mathbf{Y} \) we say \( \mathbf{X} \) is the domain and \( \mathbf{Y} \) the codomain or target space
A transformation $T: V \rightarrow W$ between vector spaces (with same scalars) $V, W$ is linear if
1) $T(u + v) = T(u) + T(v)$ for $u, v \in V$
2) $T(\alpha u) = \alpha T(u)$ for $u \in V$ and $\alpha$ a scalar

Another way of stating this is to just say
$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ for all $u, v \in V$ and all scalars $\alpha, \beta$

Shorthand: $T(u) = T\bar{u}$
$T(u + v) = T\bar{u} + T\bar{v}$

Basic idea (for homework):
take a standard basis for your vector space $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$
(represent them as column vectors). For some linear transformation $T$, let $T\bar{e}_k = \bar{a}_k$
The matrix representation $[T] = [a_1, a_2, \ldots, a_n]$