2.8 The Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \] for \( \text{Re}(s) > 1 \).

Is the Riemann Zeta Function and its properties are closely related to the distribution of primes. Bernhard Riemann was the first to study \( \zeta(s) \) with a complex variance in 1860.

We see \( \zeta(s) \) converges absolutely for \( s = \sigma + it \) with \( \sigma \geq 1 \) and \( \sigma > 1 \), indeed

\[ \sum_{n=1}^{\infty} \frac{1}{n^s} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \]

for \( \sigma > 1 \) by the Integral Test.

Riemann showed that \( \zeta(s) \) had a meromorphic continuation to all \( s \in \mathbb{C} \) with the only pole at \( s = 1 \) with residue 1.

Gamma Function

Also \( \Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \) satisfies the functional equation \( \Lambda(s) = \Lambda(1-s) \).

Here \( \Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt \) is the continuation of the factorial function with \( \Gamma(n) = (n-1)! \) for \( n \in \mathbb{Z} \) and \( x \Gamma(x) = \Gamma(x+1) \).

The \( \zeta(s) \) relates to the primes by way of the Euler Product
Claim: \( \zeta(s) = \prod_{\text{prime } p} (1 - \frac{1}{p^s})^{-1} \) for \( \Re(s) > 1 \).

Proof: \( \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \sum_{n \geq 1} \frac{1}{n^s} \)

where \( \mathbb{I} \) is the set of positive integers with prime factors \( \leq \mathbb{N} \).

We see \( \left| \sum_{n \geq 1} \frac{1}{n^s} - \sum_{n \in \mathbb{I}} \frac{1}{n^s} \right| = \sum_{n > N} \frac{1}{n^s} \to 0 \) as \( N \to \infty \).

Simple

Since \( \Gamma(x) \) has \( \nu \) poles at \( x \in \mathbb{Z}_\leq 0 \), the functional equation \( \Gamma(\frac{s}{2})\zeta(s) = 2^{s-1}\pi^{s/2}\Gamma(\frac{1-s}{2})\zeta(1-s) \)
tells us \( \zeta(s) \) has "trivial zeros" for \( s = -2, -4, -6, \ldots \).

Visualizing it

Also has series expansion by functional equation

Critical strip

Where \( \zeta(s) \) converges as a series

\( s = 0, 0 = \frac{1}{2}, 0 = 1 \)
THE RIEMANN HYPOTHESIS (CONJECTURE)

All non-trivial zeros \( \zeta(\sigma+it)=0 \) have \( \sigma=\frac{1}{2} \). Worth USD $1,000,000.
One of the millennium problems.

If RH is true then

\[
\Pi(x) = \sum_{\text{primes } \leq x} \frac{1}{\log x} + O\left(\sqrt{x} \log x\right), \quad \text{and} \quad p_{n+1} - p_n = O\left(p_n^{\frac{1}{2} + \epsilon}\right)
\]

and \( \sum_{n \leq x} \mu(n) = O\left(x^{\frac{1}{2} + \epsilon}\right) \)

CLAIM: \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \) for \( \text{Re}(s) > 1 \)

PROOF: \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n \mid m} \mu(n) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n \mid m} \mu(n) = \frac{\varphi(n)}{n^s} = \frac{1}{\zeta(s)} \)

That's why \( \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{\pi^2}{6} \) implies \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{6}{\pi^2} \).

CLAIM: \( \zeta(s-1) / \zeta(s) = \sum_{n=1}^{\infty} \phi(n) / n^s \) for \( \text{Re}(s) > 2 \)

PROOF: \( \phi(n) = n \sum_{d \mid n} \frac{\mu(d)}{d \ln d} \) so

\[
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \sum_{m \neq 1} \frac{n m^{-s}}{m^{-s} - 1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \sum_{m \neq 1} \frac{m^{-s}}{m^{-s} - 1} = \left( \sum_{m=1}^{\infty} \frac{m^{-s}}{m^{-s} - 1} \right) \left( \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \right) = \zeta(s-1) / \zeta(s).
\]
Claim: \[ \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \left(\zeta(s)\right)^2 \text{ for } \Re(s) > 1 \]

Proof: \[ \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\varphi(1)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(mn)^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)
\]

Claim: \[ \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1) \text{ for } \Re(s) > 1 \]

Proof: \[ \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\varphi(1)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(mn)^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\]

Aside: (Not in book) Convergent sums of the form \[ \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \text{ for some arithmetic function } a(n), \Re(s) > c \text{ are Dirichlet series. When } a(n) \text{ come from something interesting (Elliptic curve) we often call } \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \text{ an example of an L-series or L-function } L(s, f) \text{ where } f \text{ is the interesting thing.} \]

If \( L(s, f) \) has

1) A meromorphic continuation to all \( s \in \mathbb{C} \)
2) \( a(1) = 1 \) and \( o(a(n)) = O(n^\varepsilon) \) for \( \varepsilon > 0 \)
3) satisfies a \( s \to 1-s \) functional eq.
4) Has an Euler product.
Then we say $L(s,f)$ is in the Selberg class of L-functions, and we believe it also satisfies the Riemann Hypothesis. This is the Grand Riemann Hypothesis (GRH).

Note: It's a little more complicated than this.