CLAIM: For \( n \geq 20 \), \( \exists C > 0 \) s.t. \( \tau(n) < C \sqrt{n} \) for \( n > 1 \) \( \{18\} \)

**Proof:** \( f(n) = \frac{\tau(n)}{n^\delta} \) is multiplicative and \( f(p^j) = \frac{\phi(p^j)}{p^j} < 1 \)

For all but finitely many values for \( p, j \)

Since there exists a \( p \) large enough that \( \sqrt{p} < p^\delta \) so \( j+1 < p^\delta \) \( \forall j \)

And there exists \( j_0 \) large enough that \( (j+1) < 2^j \delta \) for all \( j > j_0 \) and so \( (j+1) < p^j \delta \) for \( j > j_0 \). The finitely many exceptions give \( C \).

\( \square \)

CLAIM: \( \sigma(n) < n \log n + n \) for \( n > 1 \)

**Proof:** \( \sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{d}{d \log d} = \sum_{d|n} \frac{1}{0} < n \log n + n \)

\( \square \)

CLAIM: \( \phi(n) > \frac{1}{4} \log n \) for \( n > 1 \)

**Proof:** \( f(n) = \frac{\phi(n)}{n^2} \) is multiplicative and \( f(p^j) = \frac{(p^j - 1)}{p^j - 1} = 1 - \frac{1}{p^j} \)

So \( f(n) \leq \prod_{p|n} \left(1 - \frac{1}{p^2}\right) \geq \prod_{p|n} \left(1 - \frac{1}{p}\right) = \frac{1}{2} \) by telescoping product \( 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \ldots \)

So \( f(n) = \frac{\sigma(n)}{n^2} \geq \frac{1}{2} \) so \( \phi(n) \geq \frac{n^2}{2 \log n} = \frac{n}{\log n} \)

WHEN \( n > 2 \). The \( n=2 \) case is easily \( \square \)

Checked.
\section{2.6 Average Orders}

Sometimes it's "of interest" to determine the magnitude, "on average" of arithmetic functions, that is \( \sum_{n \leq x} f(n) \) as \( x \to \infty \).

**Claim:** \( \sum_{n \leq x} \tau(n) = x \log x + O(x) \)

**Proof:**
\[
\sum_{n \leq x} \tau(n) = \sum_{n \leq x} \sum_{d \mid n} 1 = \sum_{d \leq x} \sum_{n \leq x/d} 1 = \sum_{d \leq x} \left[ \frac{x}{d} \right] = \sum_{d \leq x} \frac{x}{d} - \sum_{d \leq \gamma x} \frac{x}{d} = \frac{1}{\gamma} \log x + O(1) \text{ from calculus}
\]

So \( \sum_{d \leq x} \left[ \frac{x}{d} \right] = x(\log x + O(1)) + O(x) = x \log x + O(x) \)

**Note:** This technique can be improved to give \( x \log x + (2 \gamma - 1)x + O(x) \) where 
\( \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) \) is Euler's constant (Ask me how!)

Since \( \sum \log n \sim x \log x \), we have that the "average order" of \( \tau(n) \) is \( \log n \), but this is somewhat misleading as "most" \( \tau(n) \) are even smaller (see for yourself)