\( a, b \in \mathbb{P} \) \[ P_1, P_2, \ldots, P_n \] \[ \delta = \max(a, b) \]

\[ \text{CLAIM: THERE ARE INFINITELY MANY PRIMES} \]

\[ \text{PROOF: LET} \]
\[ P_1, P_2, \ldots, P_n \text{ BE A LIST OF PRIMES} \]
\[ \text{AND LET } q \text{ BE A PRIME FACTOR OF } P_1 P_2 \cdots P_n + 1, \text{ THEN} \]
\[ q \neq P_i \text{ FOR } 1 \leq i \leq n \text{ SO } q \text{ IS A PRIME NOT ON THE LIST. SO WE CAN ALWAYS FIND ANOTHER PRIME.} \]

\[ \text{IN FACT, IF } \pi(x) = \# \text{ PRIMES } \leq x \]
\[ \text{THEN } \pi(x) \sim \frac{x}{\log x} \text{ \text{PRIME NUMBER THEOREM} } \]

\[ \text{LANDAU NOTATION:} \]
1) \[ f(x) \sim g(x) \text{ MEANS } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \]
2) \[ f(x) = O(g(x)) \text{ MEANS } \exists x_0 > 0 \text{ AND } M > 0 \]
\[ \text{S.T. } \left| \frac{f(x)}{g(x)} \right| \leq M \text{ FOR } x > x_0 \]

\[ \text{EULER SHOWED } \sum_{P_k} \frac{1}{P_k} \text{ \WE'LL SHOW THIS LATER} \]
Further, it is known that
\[ \sum_{p \leq x} \frac{1}{p} = \log \log x + C + O(\frac{1}{\log x}) \] for some constant C.

**Dirichlet's Theorem for Arithmetic Progressions**

If \( a, q \in \mathbb{N} \) with \( (a, q) = 1 \) then there are infinitely many primes in the sequence \( a, a+q, a+2q, a+3q, \ldots \).

We'll touch back on this stuff in later chapters, but read §1.6 for more fun facts about primes.

**Chapter 2: Arithmetical Functions**

### §2.1 The Function \([x]\)

\([x]\) is often called the _floor function_ and \([x]\) is "the integral part of \( x \)," the largest integer less than \( x \in \mathbb{R} \). So the integer that satisfies \( x - 1 < [x] \leq x \).

Let \( \{x\} := x - [x] \in (0, 1) \) denote the "fractional part of \( x \)."
Facts about $\lceil x \rceil$:

1) $\lceil x + y \rceil \geq \lceil x \rceil + \lceil y \rceil$

2) For $n \in \mathbb{Z}$, $\lceil x + n \rceil = \lceil x \rceil + n$

3) For $m \in \mathbb{N}$, $\lfloor \frac{x}{m} \rfloor = [\frac{x}{m}]

Proof: 1) $\lceil x \rceil + \lceil y \rceil \leq x + y$ and $\lceil x \rceil + \lceil y \rceil \in \mathbb{Z}$ so $\lceil x + y \rceil \geq \lceil x \rceil + \lceil y \rceil$

2) We see $x + n - \lceil x \rceil \in \{0, 1\}$

so $x + n - 1 < \lceil x \rceil + n = x + n$, and $\lceil x \rceil + n \in \mathbb{Z}$

3) $\lceil x \rceil = mq + r$ for $q, r \in \mathbb{Z}$ and $0 \leq r < m$

so $x = mq + r + x \times 3$, so $\frac{x}{m} = q + \frac{r + x \times 3}{m}$

Since $0 \leq r + x \times 3 < m$ we have $\lceil \frac{x}{m} \rceil = q$

and $\lfloor \frac{x}{m} \rfloor = [q + \frac{r}{m}] = q = [\frac{x}{m}]$

Proposition:
The largest integer $l \geq 0$ such that $p^l | n!$ is given by $l = \sum_{i=1}^{\infty} [\frac{n}{p^i}]$

Proof: $[\frac{n}{p^i}] = \#$ of integers $\leq n$ that are divisible by $p^i$. Indeed $n = qp^i + r$ for $q, r \in \mathbb{Z}$

so $[\frac{n}{p^i}] = q$ and so $p^i, 2p^i, ..., qp^i$ are the integers divisible by $p^i$ less than $n$

Interchange sums.

So $\sum_{i=1}^{\infty} [\frac{n}{p^i}] = \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{p^i m} = \sum_{m=1}^{\infty} \frac{1}{p^i m} (\#$ of times $p^i$ appears in $m$)