THEOREM  Θ IS A QUADRATIC IRRATIONAL IF ITS CONTINUED FRACTION IS EVENTUALLY PERIODIC.

PROOF: Suppose Θ = [a₀, a₁,..., aₖ,..., aₖ⁺ₙ₋₁]

Let φ = [a₀, ..., aₖ₋₁, φₖ⁻₁, ...]

Then Θₖ = φₖ

So Θ = \( \frac{P_{k-1}φ + P_{k-2}}{Q_{k-1}φ + Q_{k-2}} \) where \( \frac{p_n}{q_n} \) is the n-th convergent of Θ.

Let \( \frac{p'_n}{q'_n} \) be the n-th convergent of φ, since φ is periodic we have φ = [aₖ,..., aₖ+n-1, φₖ⁻₁, ...] but φ is periodic so φₖ = φ so Θ = \( \frac{P'_nφ + P_{k-2}}{Q'_nφ + Q_{k-2}} \).

qₖ⁻₁ φ² + (qₖ⁻₂ - pₖ⁻₁)φ - pₖ⁻₂ = 0. Since φ ∈ Q we have that φ is a QUADRATIC. Thus Θ = x + y√d

Θ = \( \frac{P_{k-1}φ + P_{k-2}}{Q_{k-1}φ + Q_{k-2}} \) = \( \frac{(P_{k-1}φ + P_{k-2})(q_{k-1}φ + q_{k-2})}{(q_{k-1}x + q_{k-2})² - d (q_{k-1}y)²} \)

= \( \frac{x'}{y'} \) for x', y' ∈ Θ, and Θ ∈ Q so Θ is QUADRATIC.

Now suppose Θ is QUADRATIC. Consider the binary form \( F(x, y) = ax² + bxy + cy² \), \( a, b, c ∈ \mathbb{Z} \) with \( d = b² - 4ac > 0 \).

Let \( F = \left[ \begin{array}{c} p_n & q_n \\ q_{n-1} & q_{n-2} \end{array} \right] \) correspond to the
QUADRATIC $f_n(x,y) = a_n x^2 + b_n x y + c_n y^2$

SINCE $\begin{bmatrix} a_n & b_n \\ c_n & 0 \end{bmatrix}$ HAS DETERMINANT $(-1)^{n-1}$ WE SEE $f_n$ HAS THE SAME DISCRIMINANT AS $f$. Also

$a_n = f_n(1,0) = f\left(\begin{bmatrix} a_n & b_n \\ c_n & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = f(a_n, b_n)$ AND

$c_n = f_n(0,1) = f\left(\begin{bmatrix} a_n & b_n \\ c_n & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = f(a_n, b_n) = a_{n-1}$.

NOW $f(\theta, 1) = a \theta^2 + b \theta + c = 0$ so

$a_n = \frac{f_n(1,0)}{q_n^2} = \frac{f(a_n, b_n)}{q_n^2} = a_n\left(\frac{a_n^2}{q_n^2} - \theta^2\right) + b_n\left(\frac{a_n}{q_n} - \theta\right)$.

SINCE $|\theta - \frac{a_n}{q_n}| = \frac{1}{q_n^2}$ WE HAVE

$|\theta^2 - \left(\frac{a_n}{q_n}\right)^2| = |\theta - \frac{a_n}{q_n}| |\theta + \frac{a_n}{q_n}| < \frac{1 + \frac{a_n^2}{q_n^2}}{q_n^2} < \frac{2|\theta| + 1}{q_n^2}$ so

$a_n < \frac{|a| \left(2|\theta| + 1\right) + 1}{q_n^2} \Rightarrow a_n < |a| \left(2|\theta| + 1\right) + 1b_1$, so

THE $a_n$ AND $c_n$ ARE BOUNDED INDEPENDENTLY OF $n$ AND THIS SO IS $b_n = \pm \sqrt{d + 4a_n c_n}$.

BUT, FOR $n \geq 1$, $\theta = \frac{p_n \Theta_n + q_n -1}{q_n \Theta_n + q_n -1}$.

NOW $f_n(\theta_{n+1}, 1) = f\left(\begin{bmatrix} p_n \Theta_n + q_n -1 \\ q_n \Theta_n + q_n -1 \end{bmatrix}, 1\right)$

$= \frac{1}{(q_n \Theta_n + q_n -1)^2} f\left(\begin{bmatrix} p_n \Theta_n + q_n -1 \\ q_n \Theta_n + q_n -1 \end{bmatrix}, 1\right) = \frac{1}{(q_n \Theta_n + q_n -1)^2} f(\theta, 1) = 0$.

SO $\theta_{n+1}$ IS A SOLUTION TO $a_n x^2 + b_n x + c_n$ BUT THERE ARE ONLY FINITELY MANY SOLUTIONS
TO FINITELY MANY POLYNOMIALS, THUS FINITELY MANY θn, SO EVENTUALLY θl+m = θm FOR SOME l, m ∈ N, AND SO THE CONTINUED FRACTION IS PERIODIC.

§6. WRAPPING UP BAKER

PELL’S EQUATION
WE WANT TO FIND SOLUTIONS TO
\[ x^2 - dy^2 = 1, \]
WHERE d ∈ N IS NOT A SQUARE FOR x, y ∈ Z.

THIS IS CALLED PELL’S EQUATION, NAMED FOR JOHN PELL WHO HAD LITTLE TO DO WITH IT (PELL MAY HAVE FIRST USED THE OBELUS SYMBOL “÷” FOR DIVISION THOUGH) EULER WAS JUST CONFUSED.

AROUND 1650, FERMAT CONJECTURED THAT THERE WAS A SOLUTION BESIDES \((x, y) = (± 1, 0)\), AND IN THE 12TH CENTURY CE (~500 YEARS EARLIER) BHĀSKARA II (THE TEACHER) PROVED THIS USING THE CHAKRAVĀLA METHOD DUE TO BRAHMAGUPTA IN 628 CE. "WHEEL" BHĀSKARA ALSO, KIND-OF, INVENTED CALCULUS.

YOU WILL PROVE PELL’S EQUATION HAS NON-TRIVAL SOLUTIONS IN ASSIGNMENT 5.
Lagrange gave a general solution in 1768 using continued fractions.

Let the smallest \( x, y \in \mathbb{Z} \) pair solving \( x^2 - dy^2 = 1 \) be called the fundamental solution.

Theorem (Don't use in homework) Let \( \frac{p_n}{q_n} \) be the \( n \)th convergent of \( \sqrt{d} \), \( \exists \ k \in \mathbb{N} \) such that \( (p_k, q_k) \) is the fundamental solution to \( x^2 - dy^2 = 1 \).

Proof: See Chapter 8 of Baker.

Also note that we can show that if \( (x_1, y_1) \) is a fundamental solution then \( (x, y) \) is a solution iff \( x + y \sqrt{d} = (x_1 + y_1 \sqrt{d})^n \) for \( n \in \mathbb{Z} \).

Algebraic and transcendental numbers

A number \( \theta \in \mathbb{R} \) is said to be algebraic of degree \( n \) if \( \theta \) is the root of a polynomial with integer coefficients with smallest degree \( n \), \( \Phi(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \) for \( a_i \in \mathbb{Z} \).

Algebraic numbers form a ring.