CHAPTER 6: DIOPHANTINE APPROXIMATION

This chapter is mainly about approximating irrational numbers with rational numbers, often using continued fractions.

§6.1 DIRICHLET'S THEOREM

(Different from the other Dirichlet's theorem mentioned in Chapter 2)

Dirichlet's Theorem: For any $\theta \in \mathbb{R}$ and $Q \in \mathbb{N}_+$, there exist $p, q \in \mathbb{Z}$ such that

$$|q\theta - p| \leq \frac{1}{Q}.$$ 

Proof: Let $Q \in \mathbb{N}_+$. Recall $\sum_{k=1}^{\infty} x^k - [x]$. Consider the numbers $S = \{0, 1, \frac{2}{Q}, \frac{3}{Q}, \ldots, \frac{Q-1}{Q}\}$. Now consider the interval $[0, 1]$ chopped up into $Q$ sub-intervals.

$[0, \frac{1}{Q}], [\frac{1}{Q}, \frac{2}{Q}], [\frac{2}{Q}, \frac{3}{Q}], \ldots, [\frac{Q-1}{Q}, 1]$. We see $S \subseteq [0, 1]$.

Now I have $Q$ boxes $0 \frac{1}{Q} \frac{2}{Q} \cdots \frac{Q}{Q}$ in $[0, 1]$ and $Q+1$
TWO NUMBERS \( a, b \in \mathbb{S} \) MUST BE IN THE SAME BOX (Pigeonhole Principle). So \( |a - b| \leq \frac{1}{\varphi} \).

Now \( |1 - 0| = 1 \) so we can assume \( a = \frac{\varphi n + 3}{2} \) for \( 1 \leq n \leq \varphi - 1 \). So

\[
\text{IF } b = \frac{\varphi n + 3}{2} \text{ THEN } a - b = \frac{\varphi}{2} - \left[ \frac{\varphi n}{2} - \frac{\varphi}{4} \right] = \frac{(n - n')\varphi}{2} - \left[ \left( \frac{n}{2} \right) - \frac{n'}{2} \right] = \frac{q_6 - p}{2} \text{ and } |q_6 - p| \leq \frac{1}{\varphi}.
\]

\[
6 \leq q_6 \leq \varphi - 2
\]

IF \( b = 0 \) or \( 1 \) THEN \( |a - b| = \left| \frac{\varphi}{2} - \left[ \frac{\varphi n}{2} - \frac{\varphi}{4} \right] - b \right| \leq \frac{1}{\varphi}
\]

SO WE HAVE THE RESULT FOR \( \varphi \in \mathbb{N} - 1 \).

NOW SUPPOSE \( \varphi \in \mathbb{R} - 1 \) AND \( \varphi \in \mathbb{N} \), THEN \( \exists p, q \in \mathbb{Z} \) WITH \( 0 \leq q \leq \left[ \varphi \right] + 1 \) SUCH THAT \( |q_6 - p| \leq \frac{1}{\varphi} \)

\[
0 \leq q \leq \left[ \varphi \right] < \varphi. \text{ SO WE'RE DONE.}
\]

THIS TELLS US LITTLE ABOUT HOW TO FIND SUCH \( p, q \) BUT IT IS USEFUL TO KNOW A SOLUTION EXISTS.

ALSO NOTE THAT IF \( (p, q) = d \) THEN

\[
p = p'd, \quad q = q'd \quad \text{SO} \quad |q_6 - p| \leq \frac{1}{\varphi} \Rightarrow |q_6' - p'| \leq \frac{1}{\varphi d}
\]

WHICH IS A BETTER APPROXIMATION, SO WLOG WE CAN TAKE \( (p, q) = 1 \).
Corollary: There exist infinitely many pairs \( p, q \in \mathbb{Z}, q > 0 \), \((p, q) = 1 \) s.t. \( |\Theta - \frac{p}{q}| < \frac{1}{q^2} \)

iff \( \Theta \) is irrational.

Proof: Suppose \( \Theta \) is irrational, then fix \( \Phi_0 > 1 \), \( \exists p_0, q_0 \in \mathbb{Z}, 0 < q_0 < \Phi_0 \), \((p_0, q_0) = 1 \) s.t.

\[ |q_0 \Theta - p_0| = \frac{1}{\Phi_0} \Rightarrow |\Theta - \frac{p_0}{q_0}| \leq \frac{1}{q_0 \Phi_0} < \frac{1}{\Phi_0^2}. \]

Since \( \Theta \) is irrational, \( |\Theta - \frac{p_0}{q_0}| > 0 \). Let

\[ \Phi_1 > |\Theta - \frac{p_0}{q_0}|^{-1} \]

so \( \exists p_1, q_1 \in \mathbb{Z}, 0 < q_1 < \Phi_1 \) and \((p_1, q_1) = 1 \) s.t.

\[ |\Theta - \frac{p_1}{q_1}| < \frac{1}{q_1 \Phi_1} \]

and also \( |\Theta - \frac{p_1}{q_1}| < \frac{1}{q_1} |\Theta - \frac{p_0}{q_0}| \).

So \( p_1, q_1 \) is a different pair. We can do this infinitely many times \((p_1, q_1), (p_2, q_2), \ldots\).

Etc.

Now suppose \( \Theta = \frac{a}{b} \leftarrow \text{rational with } a, b \in \mathbb{Z}, b > 0. \)

Then for \( \frac{p}{q} \neq \Theta \)

\[ |\Theta - \frac{p}{q}| = \left| \frac{aq - pb}{bq} \right| \geq \frac{1}{bq} \]

so we can only FWD \((p, q)\) pairs where

\[ |\Theta - \frac{p}{q}| < \frac{1}{q^2} \text{ when } \frac{1}{q^2} > |b - \frac{p}{q}| \geq \frac{1}{bq} \text{ or } \frac{a}{b} = \frac{p}{q} \]

which is a finite set. \( \Box \)

The big idea of the above proof is that rational numbers have a best rational approximation, itself, and then we stop.