Instructions: This is not homework. This is a practice exam. It is meant to give you a sense of what the written final for MATH 228 will be like. While you are allowed to use this test however you like, I strongly encourage you try this under closed-book and timed conditions after having first reviewed and absorbed the material. If any questions surprise you or are particularly challenging, you may want to go back and review the associated material. DO NOT USE THIS AS A STUDY GUIDE, it is not comprehensive and questions on the actual exam may be completely different.

The exam has nine questions labelled 1 through 9. Each question is worth 10 marks for a total of 90 marks. There is also one optional bonus question, labelled BONUS which can recover up to five lost marks. The exam is meant to take three hours.

In the real test, to receive full credit you must explain your answers, unless otherwise stated. I recommend you try doing that for practice purposes.

Write all answers on the exam. You may use the backs of pages if necessary.

Queen's approved, non-programmable calculators are permitted on the final. Solutions are available on the course website, as are other preparation materials.

Good luck.

Student Number: SOLUTIONS

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1. —

a) Describe and sketch the set of points, \( z \), where both \( \text{Im} \, z \leq 2 \) and \( |z - 2i| < 2 \). Is this set a domain?

THE LOWER HALF OF AN OPEN DISK OF RADIUS 2, CENTERED AT 2i. THE LOWER PART OF THE BOUNDARY IS EXCLUDED BUT THE UPPER PART IS INCLUDED. THE POINTS WHERE THE TWO PARTS OF THE BOUNDARY MEET ARE EXCLUDED. IT IS NOT A DOMAIN BECAUSE THE INCLUDED BOUNDARY POINTS ARE NOT INTERIOR POINTS.

b) Describe and sketch the range of the function

\[
f(z) = \frac{8i}{z^2} - 1
\]

on the domain of definition specified in part (a). Is this range bounded?

\[
\begin{array}{c}
\text{SINCE } z \in [1, 2] \text{ AND WE SEE THAT THE CLOSEST AND FURTHEST POINTS ON THE (EXCLUDED) BOUNDARY ARE 0 AND } \\
\pm 2 + 2i, \text{ RESPECTIVELY, WE HAVE } \\
|z| \in (0, 2\sqrt{2}) \text{ SO } |z|^2 \in (0, 8) \text{ SO } \frac{8}{|z|^2} \in (0, 1) \\
\end{array}
\]

SO THE SET OF POINTS IS THE VERTICAL LINE GOING UP FROM (BUT NOT INCLUDING) \(-1\), FOREVER. IT IS NOT BOUNDED SINCE IT HAS POINTS ARBITRARILY FAR FROM THE ORIGIN.
2. For $z = x + iy$ with $x, y \in \mathbb{R}$, sketch and describe where the function \[ f(z) = \begin{cases} x^4 - ix + iy^4 + y & \text{for } |z| < 2 \\ \log(|x + iy|) + i \arg_{\mathbb{R}}(x + iy) & \text{for } |z| > 2 \end{cases} \]
is complex differentiable. Is it analytic anywhere, and if so, where?

For $|z| < 2$, we have $u(x, y) = x^4 + y$, $v(x, y) = y^4 - x$ so \[ \frac{\partial u}{\partial x} = 4x^3, \quad \frac{\partial u}{\partial y} = 1 \] so to satisfy CR's \[ 4x^3 = y^4 \] \[ 7x = y \] and $1 = -(-1) \sqrt[7]{2} \checkmark$

So the function is complex differentiable inside $|z| < 2$ on the line $x = y$ and analytic nowhere.

For $|z| > 2$, $f(z) = \sum_{n=2}^{\infty} \frac{z^n}{n^2}$ which is differentiable and analytic everywhere except for the branch cut along the negative imaginary axis.

\[ \text{Radius } 2. \]

\[ \text{Only differentiable here inside circle.} \]

\[ \text{Outside of circle, not differentiable/analytic.} \]
3. —

a) Does \( u(x, y) = x^3 + x^2 - (1 + 3x)y^2 \) have a harmonic conjugate? If so, what is one?

\[
\frac{\partial u}{\partial x} = 3x^2 + 2x - 3y^2, \quad \frac{\partial u}{\partial y} = -2(1+3x)y \\
\frac{\partial^2 u}{\partial x^2} = 6x + 2, \quad \frac{\partial^2 u}{\partial y^2} = -2 - 6x \\
\]

So \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \), so \( u \) is harmonic on all \( \mathbb{C} \), which is simply connected, so a harmonic conjugate, \( v(x,y) \), exists.

So \( v(x,y) = \int \frac{\partial u}{\partial y} \, dy = 3x^2y + 2xy - y^3 + \psi_1(x) \)

\( v(x,y) = \int -\frac{\partial u}{\partial y} \, dx = 2xy + 3x^2y + \psi_2(y) \)

So \( v(x,y) = 3x^2y - y^3 + 2xy \) works.

b) Does \( u(x, y) = xe^{x^2-y^2} \) have a harmonic conjugate? If so, what is one?

\[
\frac{\partial u}{\partial x} = e^{x^2-y^2} + 2xe^{x^2-y^2} \\
\frac{\partial u}{\partial y} = 2xe^{x^2-y^2} + 4xe^{x^2-y^2} + 4x^3e^{x^2-y^2} \\
\frac{\partial^2 u}{\partial y^2} = -2xe^{x^2-y^2} + 4xy^2e^{x^2-y^2} \\
\]

So \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{x^2-y^2}(4x+4x^3+4xy^2) \neq 0 \)

So \( u \) is not harmonic and so has no harmonic conjugate.
4. Find a branch, $F(z)$, of the multivalued function $f(z) = (z^2 - 8)^{1/3}$ that is analytic on $|z| < 2\sqrt{2}$ and $F(0) = -2$.

\[
F(z) = e^{\frac{1}{3} \log(z^2 - 8)}
\]

Since

\[
\log(z^2 - 8) = \log(-8) + \log(18 + i\pi + 2\pi i k)
\]

\[
f(0) = e^{\frac{1}{3} \log(18 + i\pi + 2\pi i k)} = e^{\frac{1}{3} \log(18)} e^{\frac{2}{3} \log(18) + \frac{2}{3} (2k + 1) \pi i / 3} \quad \text{so we need}
\]

$2k + 1$ to be a multiple of 3, $k = 1$ works.

Now $\Sigma_{2\pi}(-8)$ is analytic on $1|z - 8| < 8$ and $\Sigma_{2\pi}(-8) = \log|18| + 3\pi i$.

So letting $F(z) = e^{\frac{1}{3} \Sigma_{2\pi}(z^2 - 8)}$ works.
5. —

a) Compute the radius of convergence of the power series

\[ f(z) = \sum_{k=0}^{\infty} \sin(6ik)(z - (1 + 2i))^{3k+2} \]

and find \( f^{(5)}(1 + 2i) \).

**BY THE RATIO TEST**

\[
\left| \frac{c_{k+1}}{c_k} \right| = \left| \frac{\sin(6i(k+1)) |z - (1+2i)|^{3(k+1)+2}}{\sin(6ik) |z - (1+2i)|^{3k+2}} \right|
\]

Since \( \sin(6ik) = \frac{e^{-6ik} - e^{6ik}}{2i} \), so

\[
\left| \frac{c_{k+1}}{c_k} \right| = \left| \frac{e^{-6k-6} + e^{6k+6}}{e^{-6k} + e^{6k}} \right| |z - (1+2i)|^3
\]

As \( k \to \infty \),

\[
\lim_{k \to \infty} |z - (1+2i)|^3 < 1 \quad \text{so} \quad |z - (1+2i)|^3 < e^{-6}
\]

So \( |z - (1+2i)| < e^{-2} \)

**THE RADIUS OF CONVERGENCE IS** \( \frac{1}{e^2} \).

Since \( f(z) = \sin(6i) (z - (1+2i))^5 + \cdots \),

\[ f^{(5)}(1+2i) = 5! \sin(6i) = 120 \sin(6i) \]

b) What is the radius of convergence of the Taylor Series of

\[ f(z) = \frac{1}{z^2(z-2)^2(z^2+4)} \]

centered at \( z = 1 + i \)?

**THE POLES OF** \( f(z) \) **ARE** \( z = 0, 2, 2i, -2, i \).

**WE SEE** 0, 2, and 2i **ARE ALL EQUIDISTANT FROM** i+1 **WITH DISTANCE** \( \sqrt{2} \). **SO THE LARGEST DISK CENTERED AT** i+1 **WHERE** \( f(z) \) **IS ANALYTIC HAS RADIUS** \( \sqrt{2} \). **SO THE RADIUS OF CONVERGENCE WILL BE** \( \sqrt{2} \).
6. —

a) Evaluate the integral:

\[ \int_C \frac{5}{(z+3i)^2} \, dz \]

where \( C \) is the arc of a circle of radius 1 centered at \( z = 3i \), traversed counterclockwise from \( z = 1 + 3i \) to \( z = -1 + 3i \).

\[
\begin{align*}
C \text{ is given by } & \gamma(t) = 3i + e^{it}, \quad t \in [0, \pi]. \\
\text{If } & f(z) = \frac{5}{(z+3i)^2}, \quad f(\gamma(t)) \gamma'(t) = \frac{5}{(-3i + e^{it} + 3i)^2} (ie^{it}) \\
\int_C & \frac{5}{(z+3i)^2} \, dz = \int_0^\pi 5i e^{3it} \, dt = \left[ \frac{5}{3} e^{3it} \right]_0^\pi = \frac{5}{3} (1 - 1) = -\frac{10}{3}.
\end{align*}
\]

b) Evaluate the integral:

\[ \int \frac{1}{z^4} \, dz \]

where \( \gamma(t) = e^{t+it} \) for \( t \in [0, \frac{5\pi}{2}] \). [Hint: A picture would probably help.]
7. —

a) Evaluate the integral

\[ \int_C \frac{\log(z)}{(z-2)^6} \, dz \]

where \( C \) is a circle \( |z - 2| = 1 \) traversed three times counterclockwise.

\[ \text{BY CAUCHY'S GENERALIZED INTEGRAL FORMULA,} \quad \oint_C \frac{\log(z)}{(z-2)^6} \, dz = 3 \left( \frac{2\pi i}{5!} \frac{d^5}{dz^5} \log(z) \right) \bigg|_{z=2} \]

\[ = \frac{6\pi i}{5!} \left[ \frac{4!}{z^5} \right] \bigg|_{z=2} = \frac{6\pi i}{5} \frac{1}{32} = \frac{3\pi i}{80} \]

b) Evaluate the integral

\[ \int_\gamma z^4 e^{-1/z} \, dz \]

where \( \gamma \) is a circle \( |z - 3| = 5 \) traversed once clockwise.

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \]

So \( e^{-1/z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot z^n} = 1 - \frac{1}{z} + \frac{1}{2! \cdot z^2} - \frac{1}{3! \cdot z^3} + \frac{1}{4! \cdot z^4} - \frac{1}{5! \cdot z^5} + \ldots \)

So \( z^4 e^{-1/z} = z^4 - z^3 + \frac{z^2}{2} - \frac{z}{6} + \frac{1}{24} - \frac{1}{120} z + \ldots \)

So \( \text{Res}(z^4 e^{-1/z}, 0) = -\frac{1}{24} \)

So BY CAUCHY'S RESIDUE THM

\[ \oint_C z^4 e^{-1/z} \, dz = -2\pi i \left( -\frac{1}{24} \right) = \frac{i\pi}{60} \]
8. — Evaluate the integral

$$\int_{C} \frac{1}{1 - \cos(z)} \, dz$$

where $C$ is the snail-shaped curve illustrated below.

Now for $f(z) = \cos(z)$, $f'(z) = -\sin(z)$,

$$f''(z) = -\cos(z), \quad f'''(z) = \sin(z), \quad f^{(4)}(z) = \cos(z), \quad \text{etc.}$$

We have $\cos(z) = \sum_{n=0}^{\infty} \frac{(z^n)}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \ldots$

And $\cos(z) = \sum_{n=0}^{\infty} \frac{(z-2\pi)^{2n}}{2n!} = 1 - \frac{(z-2\pi)^2}{2} + \frac{(z-2\pi)^4}{24} - \ldots$

So $f(z) = \frac{1}{1 - \cos(z)}$ has a degree 2 pole at $z = 0$ and a degree 2 pole at $z = 2\pi$ by Cauchy's Residue Theorem:

$$\int_{C} \frac{1}{(\cos(z)^{2})} \, dz = 2\pi i \left( -3 \cos(f,0) - \text{Res}(f,0) \right)$$

$$\text{Res}(f,0) = \lim_{z \to 0} \frac{d}{dz} \frac{z^2 f(z)}{z^2 - 2z(1 - \cos(z)) - z^2 \sin(z)}$$

$$\left( \frac{1}{(1 - \cos(z))^{2}} \right)$$
\[
\lim_{z \to 0} \frac{\left(\frac{z^8}{8} - \frac{z^6}{12} + \ldots\right) - \left(\frac{z^5}{5} - \frac{z^3}{6}\right)}{z^4 \left(1 - \frac{z^2}{2^4} + \ldots\right)^2} = \lim_{z \to 0} z \left(\frac{\frac{1}{12} + \ldots}{1 - \ldots}\right) = 0.
\]

The computation is the same for \( \text{Res}(f, 2\pi) \), it's just all the \( z \)s are replaced with \( (z-2\pi) \)s. We get
\[
\text{Res}(f, 0) = \text{Res}(f, 2\pi) = 0,
\]
so the integral is just 0.
9. Let \( \phi(x, y) \) be harmonic on \( |z| < 2 \) and continuous on \( |z| \leq 2 \) with

\[
\phi(2e^{it}) = \begin{cases} 
  t^2 & \text{for } t \in (0, \pi) \\
  2\pi^2 - \pi t & \text{for } t \in (\pi, 2\pi].
\end{cases}
\]

a) Compute \( \phi(0, 0) \).

\[
\phi(re^{i\theta}) = \frac{y - r^2}{2\pi} \int_{0}^{\pi} \frac{t^2}{4 + r^2 - 4r \cos(\theta - t)} \, dt + \frac{r - r^2}{2\pi} \int_{\pi}^{2\pi} \frac{2\pi^2 - \pi t}{4 + r^2 - 4r \cos(\theta - t)} \, dt,
\]

\((\theta, 0) = 0, e^{i\theta} \Rightarrow \theta = \phi(0, 0) = 0.\)

\[
\phi(0, 0) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{t^2}{4} \, dt + \frac{1}{2\pi} \int_{\pi}^{2\pi} 2\pi^2 - \frac{\pi t}{2} \, dt
\]

\[
= \frac{1}{2\pi} \left( \frac{\pi^3}{3} \right) + \frac{1}{2\pi} \left[ (2\pi^3 - \frac{\pi^2}{2}) - (2\pi^3 - \frac{\pi^2}{2}) \right]
\]

\[
= \frac{1}{2\pi} \left[ \frac{5\pi^3}{6} \right] = \frac{5\pi^2}{12}.
\]

b) What are the max and min of \( \phi(x, y) \) on \( |z| \leq 2 \)?

\[\text{Since the max and min are attained on the boundary, and we can see } \]

\[\text{U(}2e^{i\theta}\text{) is increasing for } t \in (0, \pi] \]

\[\text{and decreasing for } t \in (\pi, 2\pi]\]

\[\text{we have the max at } t = \pi \text{ with } U(2e^{i\pi}) = \pi^2 \]

\[\text{and the min at } t = 2\pi \text{ with } U(2e^{2\pi i}) = 0.\]
BONUS: Evaluate

\[
p.v. \int_{-\infty}^{\infty} \frac{1}{(x^2 + x + 1)^2} \, dx.
\]

Roots of \(x^2 + x + 1\) are the third roots of unity (except 1) \(e^{2\pi i/3}, e^{-2\pi i/3}\)

Let \(f(z) = \frac{1}{(z^2 + z + 1)^2}\)

Using the standard trick for this

\[
p.v. \int_{-\infty}^{\infty} \frac{1}{(x^2 + x + 1)^2} \, dx = 2\pi i \text{ Res}(f, e^{2\pi i/3})
\]

Since \((z^2 + z + 1)^2 = (z - e^{2\pi i/3})^2(z - e^{-2\pi i/3})^2\), we see \(z = e^{2\pi i/3}\) is a root of degree 2, so

\[
\text{Res}(f, e^{2\pi i/3}) = \lim_{z \to e^{2\pi i/3}} \frac{1}{(z - e^{2\pi i/3})^2} \frac{d}{dz} \left( (z - e^{2\pi i/3}) f(z) \right)
\]

\[
= \lim_{z \to e^{2\pi i/3}} \frac{1}{(z - e^{2\pi i/3})^2} \frac{d}{dz} \left( z - e^{2\pi i/3} \right) = \lim_{z \to e^{2\pi i/3}} \frac{1}{(z - e^{2\pi i/3})^4} (-2(z - e^{2\pi i/3}))
\]

\[
= -\frac{2 \left( 2i \sin \left( \frac{2\pi}{3} \right) \right)}{ \left( 2i \sin \left( \frac{2\pi}{3} \right) \right)^4} = -\frac{2i \sqrt{3}}{9}
\]

So \(p.v. \int_{-\infty}^{\infty} \frac{1}{(x^2 + x + 1)^2} \, dx = \frac{4\pi \sqrt{3}}{9}\).