INSTRUCTIONS: This is not homework. This is a practice exam. It is meant to give you a sense of what the written midterm for MATH 228 will be like. While you are allowed to use this test however you like, I strongly encourage you try this under closed-book and timed conditions after having first reviewed and absorbed the material. If any questions surprise you or are particularly challenging, you may want to go back and review the associated material. DO NOT USE THIS AS A STUDY GUIDE, it is not comprehensive and questions on the actual exam may be completely different.

The exam has six questions labelled 1 through 6. Each question is worth 10 marks.

The exam is meant to take two hours.

In the real test, to receive full credit you must explain your answers, unless otherwise stated. I recommend you try doing that for practice purposes.

Write all answers on the exam. You may use the backs of pages if necessary.

Queen’s approved calculators are permitted, as they will be permitted on the actual exam.

Solutions are available on the course website, as are other preparation materials.

Good luck.

Student Number: ________________________________

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1. —Let $f(z)$ be such that

$$f(x + iy) = \begin{cases} 
-3x^2y + y^3 + i(x^3 - 3xy^2) & \text{for } |z| > 1 \\
x^5 + iy^5 & \text{for } |z| < 1.
\end{cases}$$

for $x, y \in \mathbb{R}$. Find where $f(z)$ is differentiable, sketch this region. Describe where $f(z)$ is analytic.
2. —

a) Describe and draw a rough graph the image of \( f(z) = e^z \) on the domain of definition specified by \( |\text{Im} \, z| \leq \frac{\pi}{4} \) and \( -\log(2) < \text{Re}(z) < \log(2) \).

b) Describe and draw a rough graph of the image of \( g(z) = z^3 \) on the domain of definition specified by \( \text{Arg}(z) > \frac{3\pi}{4} \) and \( |z| \geq 2 \).
3. — For what constant values of $b, c \in \mathbb{R}$ is $u(x, y) = x^3 + bx^2y + cxy^2 + y^3$ is harmonic on all $\mathbb{R}^2$? Find a harmonic conjugate, $v(x, y)$, on this region.
4. —
a) Describe and sketch the region where $f(z) = L_0(z^4 - 1)$ is analytic. Is this region a
domain? Compute $f'(z)$ in that region.

b) Find a branch of log, $L_\tau$ such that $g(z) = L_\tau(1 - z^4)$ is analytic at $z = 2$ and
$g(2) = \log(15) - 3\pi i$. 
5. — Find the partial fraction decomposition of

\[ f(z) = \frac{(2 + i)z^2 + (3 - 2i)z - (1 + 3i)}{(z - 1 - i)(z^2 - 2i)}. \]
6. —
   a) Show that if $u(x, y)$ is harmonic on a domain $D$ with harmonic conjugate $v(x, y)$ on $D$ then $u^2 - v^2$ and $2uv$ are also harmonic on $D$.

   b) Prove that $\phi(x, y) = 2(\log |x + iy|)(\text{Arg}(x + iy))$ is harmonic for $\text{Re}(z) > 0$. 