A polygonal path is just a path of line segments.

A domain is a (non-empty) open, connected subset of \( \mathbb{C} \), often denoted \( D \).

Fun fact, any polygonal path in \( D \) can be made into a path of vertical and horizontal segments in \( D \).

So thinking about \( \mathbb{C} \) as \( \mathbb{R}^2 \), coordinate pairs \((x, y)\), we can think of
\[
U : \mathbb{C} \to \mathbb{R}
\]
as a real valued function
\[
U : \mathbb{R}^2 \to \mathbb{R}, \quad u(x + iy) = u(x, y).
\]
In multivariable Calc you've seen functions like these.

**Theorem 1:** Suppose \( U : D \to \mathbb{R} \), a real-valued function \( U(x, y) \) on the domain \( D \).
(24) If $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ for all $(x, y) \in D$ then $u$ is constant in $D$.

Why? What does $\frac{df}{dx} = 0$ mean for single variable functions? Constant. Along vertical and horizontal paths $u$ is like a single-variable function, and so is constant along those paths. All points in $D$ are connected by these paths, so $u$ is constant.

Example: Suppose $\frac{\partial u}{\partial x} = y$ and $\frac{\partial u}{\partial y} = x$ for all points on $D$. Show $u = xy + c$.

Let $v(x, y) = xy$ then $\frac{\partial (u-v)}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = y - y = 0$ and $\frac{\partial (u-v)}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = x - x = 0$. So $u-v$ is constant, so $u = v + c$ so $u(x, y) = xy + c$. 

A set $S$ is **bounded** if $S$ is contained in $D(z, R)$, for some $z \in \mathbb{C}$, $R > 0$.

**Bounded.**

**Unbounded** if not.
§ 2.1 Functions of a Complex Variable

A function (or mapping) \( f : A \rightarrow B \) is a rule that assigns for every element \( a \in A \) a value \( f(a) = b \in B \). A thing that takes input and spits out outputs.

We say \( A \) is the domain of definition of \( f \) (that is, the set where \( f \) makes sense) and the set of \( f(a) \)'s is called the range of \( f \).

Note: The "domain of definition" does not have to be a "domain" as defined in §1.6, which is an open connected set.

We are accustomed to functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) like \( f(x) = x^2 + 2 \), or functions \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) like \( f(x, y) = x^2 - y^2 \) or \( f(t) = [t, t^2, t^3] \).
We want to talk about functions where \( f: \mathbb{C} \rightarrow \mathbb{C} \) or \( f: A \rightarrow B \) where \( A, B \subseteq \mathbb{C} \).

Not so strange at first. Let \( f(z) = \frac{z^5}{z^3 - 1} \)
we see it takes complex numbers
to other complex numbers: \( f(0) = \frac{0}{0 - 1} = 0 \)
\( f(i) = \frac{i^3}{i^3 - 1} = \frac{-i}{-1 - 1} = \frac{i}{1 + 1} = \frac{i(1 - i)}{2} = \frac{1 - i}{2} \). etc.

What is the domain of definition for \( f(z) \)?
All of \( \mathbb{C} \) except where \( z^3 - 1 = 0 \), since we
don't divide by zero. These are the cube roots
of unity: \( \pm 1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}} \).

For more general \( f: \mathbb{C} \rightarrow \mathbb{C} \) let
\( u(z) = \text{Re}(f(z)) \)
and \( v(z) = \text{Im}(f(z)) \). So
\( u, v: \mathbb{C} \rightarrow \mathbb{R} \), and
\( f(z) = u(z) + iv(z) \)
\( f: \mathbb{C} \rightarrow \mathbb{C} \)
\( f(x + iy) = u(x + iy) + iv(x, y) \)
\( f: \mathbb{R}^2 \rightarrow \mathbb{C} \)
\( f(x, y) = (u(x, y), v(x, y)) \)
\( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

So we see that functions from \( \mathbb{C} \rightarrow \mathbb{C} \) are
also the functions \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

Example: Write \( f(z) = \frac{1}{z} \) as
\( f(x, y) = u(x, y) + iv(x, y) \).