A sequence of functions, 

\[ f_n(z) \]  

is said to converge uniformly to a function \( f(z) \) on the set \( T \) if  

\[ |f_n(z) - f(z)| < \varepsilon \]  

for all \( z \in T \) if \( n \) is big enough.

We say \( \sum_{j=0}^{\infty} f_j(z) \) converges uniformly on \( T \) if the sequence of partial sums, \( s_n(z) = \sum_{j=0}^{n} f_j(z) \), converges uniformly on \( T \).  

We like uniform convergence because it preserves: continuity, analyticity, derivatives, integrals, etc.

We are distinguishing from pointwise convergence.

§5.2 Taylor Series

We've seen Taylor form for polynomials, now we have it for all analytic functions.

Definition 3: If \( f \) is analytic on the open disk \( |z-z_0| < r \) then
\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \ldots \]

Converges on the open disk and converges uniformly on any smaller disk \(|z-z_0| \leq r\) where \(r < R\).

This is called the Taylor series expansion of \(f(z)\). If \(z_0 = 0\) it is called the Maclaurin series for \(f(z)\). \(R\) is the radius of convergence.

Why? \[ f(z) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{w-z} \, dw \quad \text{and} \quad \frac{1}{w-z} = \frac{1}{w-z_0} - \frac{1}{w-z_0} \frac{1}{1 - \frac{(z-z_0)}{w-z_0}} = \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left( \frac{z-z_0}{w-z_0} \right)^n \]

so \[ f(z) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{w-z} \, dw = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \left( \frac{z-z_0}{w-z_0} \right)^n \right|_{C_r} \]

Interchange order of summation and integration because of uniform convergence.

Examples: Find the Maclaurin series expansion of \(\log(1+z)\) and radius of convergence.

The largest disk we can make around \(z_0 = 0\) where \(\log(1+z)\) is analytic is radius 1, so that's where series converges, \(|z| < 1\).
\[ \frac{d \log(1-z)}{dz} = -\frac{1}{1-z} = -(1-z)^{-1} \]

\[ \frac{d^2 \log(1-z)}{dz^2} = -(1-z)^{-2} \]

\[ \frac{d^3 \log(1-z)}{dz^3} = -2(1-z)^{-3} \]

\[ \vdots \]

\[ \frac{d^n \log(1-z)}{dz^n} = -(n-1)! (1-z)^{-n} \]

So, \( a_n = \frac{d^n \log(1-z)}{dz^n} \bigg|_{z=0} = -(n-1)! \). For \( n = 1, 2, 3, \ldots \) and \( a_n \log(1-z) = 0 \)

\[ \log(1-z) = \sum_{n=1}^{\infty} \frac{-(n-1)!}{n!} (1-z)^n = \sum_{n=0}^{\infty} \frac{-z^n}{n} \]

**Theorem 4:** If \( f(z) \) is analytic on \( |z-z_0| < r \)

Then the Taylor series of \( f'(z) \) can be obtained by term-by-term differentiation with same radius of convergence.

**Example:** From previous example,

\[ -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \]

So, \( \frac{1}{1-z} = \frac{d}{dz} \left( -\log(1-z) \right) = \frac{d}{dz} \sum_{n=1}^{\infty} \frac{z^n}{n} \]

\[ = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n} = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=0}^{\infty} z^n \]

For \( |z| < 1 \).

**Example** \( e^z \) is analytic everywhere, thus

radius of convergence is infinite, also