**Dirichlet Problem:** Find a function \( \phi(x,y) \) continuous on a domain \( D \) and its boundary, harmonic in \( D \), and taking specified values on the boundary of \( D \).

To solve this, we use the following theorem.

**Theorem 28:** Let \( \phi_1(x,y) \) and \( \phi_2(x,y) \) each be harmonic in a bounded domain \( D \) and continuous on \( D \) and its boundary. Furthermore, suppose \( \phi_1 = \phi_2 \) on the boundary of \( D \), then \( \phi_1 = \phi_2 \) on \( D \).

Why? If \( \phi_1 = \phi_2 \) on the boundary of \( D \), then \( \phi_1 - \phi_2 = 0 \) on the boundary of \( D \) and is harmonic on \( D \). By the Max-Min Principle 0 is the max and min of \( \phi_1 - \phi_2 \) on \( D \), so \( \phi_1 - \phi_2 = 0 \) on \( D \), thus \( \phi_1 = \phi_2 \) on \( D \).
WE HAVE ALREADY SOLVED DIRICHLET'S PROBLEM IN SOME Instances USING THIS UNIQUENESS OF Solution FOR UNBOUNDED DOMAINS

BUT THERE ARE GENERAL SOLUTIONS TO DIRICHLET'S PROBLEM ON A DISK

POISSON'S INTEGRAL FORMULA

THM 30: LET U BE A REAL-VALUED FUNCTION DEFINED ON THE CIRCLE C_R: \(|z| = R\), AND CONTINUOUS EXCEPT FOR POSSIBLY Finitely-MANY DISCONTinuities. THEN

\[ u(r e^{i\theta}) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{u(r e^{i\phi})}{r^2 + r^2 - 2rr\cos(\theta - \phi)} d\phi \]

IS HARMONIC FOR \(|r e^{i\theta}| < R\) AND CONTINUOUS UP TO BOUNDARY WHERE THE BOUNDARY IS CONTINUOUS. (WHY? CAUCHY'S THEOREM IS.)

EXAMPLE: SUPPOSE ON \(|z| = 3\) WE HAVE U AS IN THE PICTURE

\[ u = 2 \quad 2\pi/3 \quad u = 1 \quad 2\pi/3 \quad u = 3 \quad u = 1 \quad -2\pi/3 \quad -2\pi/3 \]

FIND HARMONIC \(u(r e^{i\theta})\) ON \(|z| < 3\) THAT IS CONTINUOUS UP TO BOUNDARY
\[
\nu(r e^{i\theta}) = \frac{1}{2\pi} \left( \int_0^{2\pi/3} \frac{1}{\sqrt{9 + r^2 - 6r \cos(t - \theta)}} dt + \int_{2\pi/3}^{4\pi/3} \frac{3}{\sqrt{9 + r^2 - 6r \cos(t - \theta)}} dt \right)
\]

So, for example, \( \phi(0, 0) = \phi(0 e^{i0}) \) so \( r = 0 \) and \( \theta = 0 \) so
\[
\phi(0, 0) = \frac{1}{2\pi} \left( \int_0^{2\pi/3} \frac{1}{9} dt + \int_{2\pi/3}^{4\pi/3} \frac{3}{9} dt + \int_{4\pi/3}^{2\pi} \frac{5}{9} dt \right)
\]

= 2.

Other, more difficult integrals can be evaluated using other techniques (Wolfram Alpha).
Chapter 5: Series Representations for Analytic Functions

Section 5.1 Sequences and Series

We already know what sequences are, 
\[ z_n = 1 + \frac{i}{n} \] 
so 
\[ \lim_{n \to \infty} z_n = 1. \]

We have seen this in calculus. We will now remind you of series, which also make sense for \( \mathbb{C} \).

Definition: A series is a formal expression of the form:

\[ s = \sum_{j=0}^{\infty} a_j = a_0 + a_1 + a_2 + \cdots \]

where the \( a_j \)'s, we let

\[ s_n = \sum_{j=0}^{n} a_j \]

be the \( n \)th partial sum of \( s \) and 

if 
\[ \lim_{n \to \infty} s_n = L \]
for some \( L \in \mathbb{C} \), then 
we say 
\[ \sum_{j=0}^{\infty} a_j = L. \]
**Example:** The series \( \sum_{k=0}^{\infty} c^k \) converges to \( \frac{1}{1-c} \) if \( c \) is less than 1. Called **geometric series**.

Why? \( S_n = \sum_{k=0}^{n} c^k = 1 + c + c^2 + \ldots + c^n = \frac{c^{n+1} - 1}{c - 1} \) (unless \( c = 1 \)) = \( \frac{1 - c^{n+1}}{1-c} \).

So \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - c^{n+1}}{1-c} = \frac{1 - \lim_{n \to \infty} c^{n+1}}{1-c} \).

If \( |c| > 1 \) then \( c^{n+1} \to \infty \), if \( |c| < 1 \) then \( c^{n+1} \to 0 \).
If \( |c| = 1 \) then \( c^{n+1} \) spins around the circle as \( n \to \infty \) and if \( c = 1 \) the series is just \( \sum_{k=0}^{\infty} 1 = 1 + 1 + 1 + \ldots \) which is clearly divergent.

We have similar tests for series as in calculus:

**Thm 1 (Comparison Test):** Suppose for \( \sum_{s=0}^{\infty} a_s \), \( |a_s| \leq M_s \) for all \( s \)

Then if \( \sum_{s=0}^{\infty} M_s \) converges, so does \( \sum_{s=0}^{\infty} a_s \).
**THM 2.** *(Ratio Test)* If for \( s = \sum_{m=0}^{\infty} a_m \)

have

\[
\lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = L \quad \text{as} \quad m \to \infty
\]

then \( s \) converges if \( L < 1 \) and

diverges if \( L > 1 \) (inconclusive if \( L = 1 \)).

*Why? Triangle Inequality, Mostly.*

**Example:** Show \( \sum_{j=1}^{\infty} \frac{1+2i^j}{j^5} \) converges

Comparison Test

\[
\left| \frac{1+2i^j}{j^5} \right| \leq \frac{|1| + |2i^j|}{|j|^5} = \frac{1 + 2}{j^5} = \frac{3}{j^5}
\]

Since \( \sum_{j=1}^{\infty} \left( \frac{1}{j^5} + \frac{2}{j^5} \right) = \sum_{j=1}^{\infty} \frac{1}{j^5} + 2 \sum_{j=1}^{\infty} \frac{1}{j^5} \)

both converge by

\( p \)-Series Test \( \left( \sum_{j=1}^{\infty} \frac{1}{j^p} \text{ converges if } p > 1 \right) \)

So \( \sum_{j=1}^{\infty} \frac{1+2i^j}{j^5} \) converges

**Example:** Show \( \sum_{k=0}^{\infty} \frac{(1+2i)^k}{k!} \) converges

Ratio Test: \( a_k = \frac{(1+2i)^k}{k!} \)

\[
\left| \frac{a_{k+1}}{a_k} \right| = \frac{k! |1+2i|^{k+1}}{(k+1)! |1+2i|^k} = \frac{|1+2i|}{k+1} \to 0 \quad \text{so}
\]

it converges.