\[ \frac{z_1}{z_2} = \frac{z_1 \overline{z}_2}{|z_2|^2} \in \text{real denominator}. \]

### 1.3 Vectors And Polar Forms

We can think of complex numbers as points on the plane, and so we can also think of them as vectors.

And since we already saw that that addition is coordinate-wise, we have that the addition of complex numbers is the addition of vectors. Also the length of the vector is just the magnitude \(|z|\).

These observations give us the **triangle inequality**: \(|z_1 + z_2| \leq |z_1| + |z_2|\).

Because the length of a side of a triangle is no greater than the sum of the lengths of the remaining side.

Can also see: \(|z_1| + |z_2 - z_1| \geq |z_2 + (z_2 - z_1)| = |z_2|\)

Or \(|z_2 - z_1| \geq |z_2| - |z_1|\).
Instead of \( z = x + iy \) (rectangular or Cartesian coordinates) we can use polar coordinates:
\[
x = r \cos \theta, \quad y = r \sin \theta, \quad r > 0, \quad \theta \in \mathbb{R}
\]
where \( r = |z| \) and \( \theta \) is the angle \( z \) makes with the positive \( x \)-axis.

\[
z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)
\]

**Example:** \( z = 3 + 4i \) then \( |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \)

\[
\tan \theta = \frac{4}{3}, \quad \theta = \arctan\left(\frac{4}{3}\right)
\]

So \( 3 + 4i = 5(\cos \theta + i \sin \theta) \) where \( \theta = \arctan\left(\frac{4}{3}\right) \)

or \( \arctan\left(\frac{4}{3}\right) + 2\pi \)

\( \arctan\left(\frac{4}{3}\right) + 4\pi \)

\( \arctan\left(\frac{4}{3}\right) - 2\pi \)

The set of all possible values of \( \theta \) is called the *argument* of \( z \), or \( \text{arg} \ z \).
Once we have one possible value for $\theta$, say $\theta_0$, then

$$\arg z = \theta_0 + 2\pi k \quad | k = 0, \pm 1, \pm 2, \ldots \frac{\pi}{2} \quad (k \in \mathbb{Z})$$

We have to be careful when computing $\theta$ because the range of $\arctan \theta$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so if $z = -3 - 4i$ then

$$\theta = \arctan \left(\frac{-4}{-3}\right) + \pi$$

Because we are in the third quadrant.

Also, if $\text{Re}(z) = 0$ then $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

\[\theta = \frac{\pi}{2} + 2\pi k \quad \text{and} \quad \theta = \frac{3\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}\]

Often we want to think of argument as a well-defined function. Let

$$\text{Arg } z = \Theta \in \arg z \cap (-\pi, \pi]$$

That is the value of $\Theta$ between $-\pi$ and $\pi$.

Call this the **principal value of argument**.
This gives rise to the concept of a branch cut.

Arg(z) is continuous everywhere except here.

\[ z \in (-\infty, 0] \]

There was nothing special about this line; we chose it by convention. Let \( \text{Arg}_\pi z = \theta \Longleftrightarrow \text{Arg} z \in \left( -\pi, \pi + 2\pi \right] \).

This is another branch of argument.

Let \( \pi = \frac{3\pi}{4} \).

Line of discontinuity, branch cut.

\[ \arg(z_2) = \pi \]

\[ \arg(z_3) = \frac{3\pi}{2} \]

Branch cut

Doesn't even need to be a straight line, but we won't worry about that; note however that

\[ \text{Arg}(z) = \text{Arg}_\pi (z) \]

Notation: \( \text{cis} (\theta) = \cos \theta + i \sin \theta, \quad \theta \in \mathbb{R} \).

Example: Write \(-1 + \sqrt{3}i\) in polar form using \( \text{Arg}_\pi \).
\[ |-1 + i\sqrt{3}| = \sqrt{1 + 3} = 2 \quad \text{so} \quad r = 2 \]

\[ \phi = \arctan \left( \frac{\sqrt{3}}{1} \right) = \frac{\pi}{3} \]

\[ \text{so} \quad \arg \left( -1 + i\sqrt{3} \right) = 3\pi - \frac{\pi}{3} = \frac{8\pi}{3} \]

\[ \text{so} \quad -1 + i\sqrt{3} = 2 \cis \left( \frac{8\pi}{3} \right) \]

**Now a fun observation about multiplication**

Suppose \( Z_1 = r_1 \cis \theta_1 \) and \( Z_2 = r_2 \cis \theta_2 \)

\[ Z_1 Z_2 = r_1 r_2 \left( \cos \theta_1 + i \sin \theta_1 \right) \left( \cos \theta_2 + i \sin \theta_2 \right) \]

\[ = r_1 r_2 \left[ \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right] \]

\[ = r_1 r_2 \cis(\theta_1 + \theta_2) \]

**So, the modulus of the product is the product of the moduli:**

\[ |Z_1 Z_2| = |Z_1| |Z_2| \]

**And the argument of the product is the sum of the arguments:**

\[ \arg Z_1 Z_2 = \arg Z_1 + \arg Z_2 \] (look familiar)

Pairwise addition of sets.
13) So

Geometrically:

\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis}(\theta_1 - \theta_2) \]

Similarly, \[ \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \text{ cis} \arg z_1 - \arg z_2 \]

That is, \[ \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \text{ AND } \arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \]

Example: Write \( \frac{2-2i}{-1+i \sqrt{3}} \) in polar form

We already know \(-1+i\sqrt{3} = 2 \text{ cis}(\frac{5\pi}{3})\)

Also

\[ \frac{2-2i}{-1+i \sqrt{3}} = \frac{2 \sqrt{2}}{2} \text{ cis} \left( -\frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right) = \sqrt{2} \text{ cis} \left( \frac{3\pi}{12} \right) = \sqrt{2} \text{ cis} \left( \frac{\pi}{12} \right) \]

\[ \S 1.4 \text{ The complex exponential} \]

We want to define \( e^z = e^{x+iy} \)

Presumably, we want \( e^{x+iy} = e^x e^{iy} \) and we know what \( e^x \) is already.