We say a domain $D$ is **simply connected** if every loop in $D$ is C.D. to a point.

**Theorem 9:** (Cauchy's Integral Theorem)

If $f$ is analytic on an S.C.-domain $D$,

$$\oint_{\Gamma} f(z) \, dz = 0$$

for any closed loop $\Gamma$ in $D$.

**Theorem 8:** (Deformation Invariance Thm)

If $f$ is analytic in a domain $D$ and $\Gamma_0 \sim \Gamma_1$ in $D$ then

$$\oint_{\Gamma_0} f(z) \, dz = \oint_{\Gamma_1} f(z) \, dz$$

Why?! Well, it's a very complicated answer.
It uses Green's theorem and the Cauchy-Riemann eqs, and the fact that the range of $\mathbf{z}(s,t)$ is simply-connected.

If you really want to know why, see section 4.4 of the book or ask me sometime.

Example: Compute \[ \int_{\Gamma} \frac{5z+1}{(z-1)(z+2)} \, dz \] where \( \Gamma \) is the path shown.

We can see $\frac{5z+1}{(z-1)(z+2)}$ is analytic everywhere except $z=1, -2$, so we can deform the curve on the domain containing it.

\[
\int_{\Gamma} \frac{5z+1}{(z-1)(z+2)} \, dz = \int_{\gamma_1} \frac{5z+1}{(z-1)(z+2)} \, dz + \int_{\gamma_2} \frac{5z+1}{(z-1)(z+2)} \, dz + \int_{\gamma_3} \frac{5z+1}{(z-1)(z+2)} \, dz
\]
Now $Y_2$ is a closed loop contained in an S.C. domain where $\frac{5z+1}{(z-1)(z+2)}$ is analytic.

So $\int \frac{5z+1}{(z-1)(z+2)} \, dz = 0$. Also $\frac{5z+1}{(z-1)(z+2)} = \frac{2}{z-1} + \frac{3}{z+2}$.

So $\int \frac{5z+1}{P(z-1)(z+1)} \, dz = \int \frac{2}{z-1} \, d\tau + \int \frac{2}{z+1} \, d\sigma + \int \frac{5}{z+2} \, d\tau + \int \frac{5}{z+2} \, d\tau$

$= 2(2\pi i) + 2(0) + 3(0) + 3(2\pi i)$

$= 10\pi i$.

§4.5 Cauchy's Integral Formula

A contour is **simple** if it doesn't self-intersect except possibly at endpoints.

**Theorem 14**

Cauchy's Integral Formula: Let $P$ be a simple closed positively oriented contour. If $f$ is analytic on a S.C. domain containing $P$ and $z_0$ is in the interior of $P$ then

$$f(z_0) = \frac{1}{2\pi i} \int_P \frac{f(z)}{z-z_0} \, dz$$
IDEA:

\[ \frac{1}{2\pi i} \int_{|z-z_0|=r} f(z) \, dz = z_0. \]

WHY? EASIER TO EXPLAIN WITH TAYLOR SERIES BUT...

\[ \int_{|z-z_0|=r} \frac{f(z)}{z-z_0} \, dz = \int_{\Gamma} \frac{f(z)}{z-z_0} \, dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} \, dz \]

CAN REPLACE \( \Gamma \) WITH CR A CIRCLE OF RADIUS \( r > 0 \) CENTERED AT \( z_0 \) BY CAUCHY'S INTEGRAL THEOREM SO

\[ \int_{|z-z_0|=r} \frac{f(z)}{z-z_0} \, dz = 2\pi i f(z_0) + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} \, dz \]

BUT \[ \left| \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} \, dz \right| \leq \lambda(C_r) \max_{z \in C_r} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| \]

\[ \leq (2\pi r) \frac{M_r}{r} = 2\pi M_r \]

WHERE \( M_r = \max_{z \in C_r} |f(z) - f(z_0)| \)
\[ \lim_{r \to 0} M_r = 0 \quad \text{so} \quad \lim_{r \to 0} \oint_C \frac{f(z)}{z - z_0} \, dz = 0. \]

But the integral is the same for all \( C \), so
\[ \oint_C \frac{f(z) - f(z_0)}{z - z_0} \, dz = 0. \]

So
\[ \oint_C \frac{f(z)}{z - z_0} \, dz = \oint_C \frac{f(z_0)}{z - z_0} \, dz = 2\pi i f(z_0). \]

**Example:** Evaluate \( \oint_C \frac{e^{z^2}}{z^2 - 1} \, dz \) counterclockwise.

\[ \frac{e^{z^2}}{z^2 - 1} \, dz = \frac{1}{2} (2\pi i) e^{\frac{(i)^2}{2}} = \frac{1}{2} (2\pi i) e^{-\pi i/2}. \]

\[ |z^2 - 1| = 1 \quad \text{continuous \( \epsilon \)-deformed circle, \( \epsilon \)-analytic around.} \]

**Example**
\[ \oint_{|z| = 1} \frac{z^2 + 1}{z(z - 2)} \, dz = \oint_{|z| = 1} \frac{(z^2 + 1)/(z - 2)}{z} \, dz. \]

\[ = 2\pi i \left( \frac{(0^2 + 1)}{(0 - 2)} \right) = 2\pi i \left( -\frac{1}{2} \right) = -\pi i. \]

Since \( f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} \, dw \).
We are tempted to say
\[ f'(z) = \frac{d}{dz} \int_{\gamma} \frac{f(w)}{w-z} \, dw = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^2} \, dw \]

\[ = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^2} \, dw. \text{ Indeed, this works and we can say something stronger} \]

**Theorem 15.** Let \( g(z) \) be continuous on the contour \( \gamma \) (that's it) let \( \gamma \) (not necessarily closed)

\[ g(z) = \int_{\gamma} \frac{g(w)}{w-z} \, dw \text{ for } w \neq \gamma \]

"Not \( \gamma \)"

Then \( g \) is analytic for all \( z \neq \gamma \).

Why the heck should that be true?!

We can show \( \lim_{\Delta z \to 0} \frac{G(z+\Delta z)-G(z)}{\Delta z} = \int_{\gamma} \frac{g(w)}{(w-z)^2} \, dw \)

explicitly (see book).

But wait... if \( f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^2} \, dw \)

Then since \( \frac{f(w)}{(w-z)} \) is continuous for \( w \neq \gamma \)

For \( \gamma \) that means \( f'(z) \) is analytic.