43. WILL NOT WORK FOR CLOSED CURVE AROUND ORIGIN:

\[ \gamma \]

\[ \mathbb{R} \times \mathbb{R} \]  \( \times \)  AN ANTI-DERIVATIVE ON A DOMAIN CONTAINING \( \gamma \) BECAUSE OF THE BRANCH CUT.

**Corollary 2:** IF \( f(z) \) IS CONTINUOUS ON \( D \) WITH ANTI-DERIVATIVE \( F(z) \) ON \( D \) THEN \( \int_{\gamma} f(z) \, dz \) FOR ALL CLOSED CONTOURS \( \gamma \).

**Why?** \[ \int_{\gamma} f(z) \, dz = F(z_1) - F(z_0) = 0 \]  \( \text{since} \ z_1 = z_0 \text{ for closed } \gamma \).

IN FACT, WE CAN SAY SOMETHING STRONGER

**Theorem 7:** LET \( f \) BE CONTINUOUS IN A DOMAIN \( D \). THEN THE FOLLOWING ARE EQUIVALENT.
a) \( f \) has an antiderivative on \( D \).

\[ \int \]

b) \( \oint_{C} f(z) \, dz = 0 \) for all closed contours \( C \).

\[ \int \]

c) If \( \Gamma_1 \) and \( \Gamma_2 \) have the same endpoints and direction, \( \oint_{\Gamma_1} f(z) \, dz = \oint_{\Gamma_2} f(z) \, dz \) (path-independence).

\[ \int \]

Why? Already showed a) \( \Rightarrow \) b) in Corollary 2. b) \( \Rightarrow \) c) because \( \Gamma_1 - \Gamma_2 \) is a closed loop.

\[ \oint_{\Gamma_1 \setminus \Gamma_2} f(z) \, dz = 0 \]

\[ \int \]

\[ \oint_{\Gamma_1} f(z) \, dz - \oint_{\Gamma_2} f(z) \, dz = 0 \Rightarrow \oint_{\Gamma_1} f(z) \, dz = \oint_{\Gamma_2} f(z) \, dz \]

\[ \int \]

c) \( \Rightarrow \) b) because if \( \gamma \) is any curve from \( z_0 \) to \( z \) then we can show \( F(z) = \int_{z_0}^{z} f(w) \, dw \) is an antiderivative of \( f(z) \). [See book].
§4.4 CAUCHY'S INTEGRAL THEOREM

We say the loop $\Gamma_0$ is continuously deformable (CD) to $\Gamma_1$ in the domain $D$ if you can bend $\Gamma_0$ to be $\Gamma_1$ while staying in $D$.

We can think of $\Gamma_s$ set $s \in [0,1]$ as the set of curves in between, each with parametrization $\Gamma_s(t)$ with $t \in [0,1]$.

**Definition 5**: $\Gamma_0$ is CD to $\Gamma_1$ in $D$ if there exists continuous $Z(s,t) = \Gamma_s(t)$ for $s, t \in [0,1]$ and $Z(s,t) \subset D$.

**Example**: $\Gamma_5 \sim \Gamma_4$ but $\Gamma_1$ is not CD to $\Gamma_2$, $\Gamma_3$, or $\Gamma_4$. 
We say a domain $D$ is \underline{simply connected} if every loop in $D$ is connected to a point.

\begin{itemize}
  \item Simply connected
  \item S.C.
  \item Not S.C.
  \item Not S.C.
\end{itemize}

**Theorem 9:** (Cauchy's Integral Theorem)
If $f$ is analytic on an S.C. domain $D$

\[
\oint_{\Gamma} f(z) \, dz = 0
\]

For any closed loop $\Gamma$ in $D$

**Theorem 8:** (Deformation Invariance Thm)
If $f$ is analytic in a domain $D$ and $\Gamma_0 \sim \Gamma_1$ in $D$ then

\[
\oint_{\Gamma_0} f(z) \, dz = \oint_{\Gamma_1} f(z) \, dz
\]

Why? Well, it's a very complicated answer.
IT USES GREEN'S THEOREM, AND THE CAUCHY-RIEMANN Eqs., AND THE FACT THAT THE RANGE OF $z(s,t)$ IS SIMPLY-CONNECTED.

IF YOU REALLY WANT TO KNOW WHY, SEE SECTION 4.4 OF THE BOOK OR ASK ME SOMETIME.

**EXAMPLE:** COMPUTE $\int_{\Gamma} \frac{5z+1}{(z-1)(z+2)} \, dz$ WHERE

![Diagram](image)

WE CAN SEE $\frac{5z+1}{(z-1)(z+2)}$ IS ANALYTIC EVERYWHERE EXCEPT $z=1, -2$, SO WE CAN DEFORM THE CURVE ON THE DOMAIN CONTAINING IT

$$\int \frac{5z+1}{(z-1)(z+2)} \, dz = \int_{z_1} \frac{5z+1}{(z-1)(z+2)} \, dz$$

$$+ \int_{z_2} \frac{5z+1}{(z-1)(z+2)} \, dz$$

$$+ \int_{z_3} \frac{5z+1}{(z-1)(z+2)} \, dz$$