We can compute the length of a smooth curve with the integral formula:

\[ l(\gamma) = \text{length of } \gamma = \int_a^b \frac{ds}{dt} dt \]

where \( \frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2} = |\gamma'(t)| \)

So in the previous example, the length of \( \gamma_2 \) is:

\[ \gamma_2(x) = 1 + i + e^{i(\pi/2-\pi t)} \Rightarrow \gamma_2'(t) = -i\pi e^{i(\pi/2-\pi t)} \]

So \( |\gamma_2'(t)| = \pi \).

So \( l(\gamma) = \int_0^1 \pi dt = \pi \), which is what we expect.

The length of a contour is just the sum of the lengths of the curves:

\[ l(\Gamma) = \text{length of } \Gamma = l(\gamma_1) + l(\gamma_2) + \ldots + l(\gamma_n). \]

A simple closed contour is a contour that starts where it ends but otherwise does not self-cross. Ex: \( \square \) not \( \bigcirc \) this \( \bigcirc \) not this.
84) **Jordan Curve Theorem**

Any simple closed contour separates $C$ into two domains, each having a curve as a boundary. One domain is bounded, called the interior, and one is not, called the exterior.

![Diagram](Exterior_Interior)

Very intuitive theorem, not easy to prove.

If the curve is oriented such that the interior is on the left of the curve (counter-clockwise), we say it is positively oriented. Otherwise negatively oriented.

![Diagram](Positively_Oriented_Negatively_Oriented)
§4.7 CONTOUR INTEGRALS

Idea: Given a curve $\gamma$ and a function $f(z)$ on a region of $\mathbb{C}$ containing $\gamma$, we can chop up the curve into pieces and evaluate $f(z)$ at points on each piece.

Let

$$S_n = f(c_1)(z_1 - z_0) + f(c_2)(z_2 - z_1) + \cdots + f(c_n)(z_n - z_{n-1})$$

looks like

$$\sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k$$

Definition 3: We say $f$ is integrable along $\gamma$ if, taking partitions where $n \to \infty$ and $\Delta z_k \to 0$, then $S_n$ always goes to some $L \in \mathbb{C}$.

We say

$$L = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta z_k = \int_{\gamma} f(z) \, dz$$

From this we have some of the obvious rules:
\[ \int g(x) \, dx = \int f(x) \, dx \pm \int h(x) \, dx \]

\[ \int cf(x) \, dx = c \int f(x) \, dx \text{ for } c \in \mathbb{C} \]

\[ \int f(x) \, dx = -\int f(x) \, dx \]

**Theorem 2:** If \( f \) is continuous on the directed smooth curve \( \gamma \), then \( f \) is integrable on \( \gamma \).

**How do we evaluate** \( \int_{\gamma} f(x) \, dx \)?

Well, how do you evaluate \( \int_{a}^{b} f(t) \, dt \) when \( f \) is complex valued?

\( f(t) = u(t) + iv(t) \)

\( \text{(} \gamma \text{ is just the real line segment } [a,b]) \)

\( \text{REAL-VALUED FUNCTIONS} \)

\[ \int_{a}^{b} f(t) \, dt = \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt \]

\[ = [u(t)]_{a}^{b} + i [v(t)]_{a}^{b} = [u(t) + iv(t)]_{a}^{b} \]

\( \text{AT I: DERIVATIVES of } u(t) \text{ and } v(t) \)

Let \( F(t) = u(t) + iv(t) \), then \( F'(t) = u'(t) + iv'(t) \)

\[ = u(t) + iv(t) \]

\[ = f(t). \]