CHAPTER 4: COMPLEX INTEGRATION

89.1 Contours: You've seen parametric curves before, \[ \gamma(t) = [\cos(t), \sin(t)] = \gamma(t), \quad t \in [0, 2\pi] \]

We can write \[ \gamma(t) = [x(t), y(t)] \]

as \[ \gamma(t) = x(t) + iy(t), \quad t \in [a, b] \]

This is called the parameterization of \( \gamma \).

We say a contour is smooth if it has no corners and doesn't cross itself.

**Definition 1:** A point set \( \gamma \) in \( \mathbb{C} \) is said to be a smooth arc if it is the range of some complex-valued function \( z(t) = x(t) + iy(t) \) for \( t \in [a, b] \) where

1) \( z(t) \) has continuous derivative \( z'(t) = x'(t) + iy'(t) \) on \( [a, b] \).

2) \( z'(t) \) never vanishes on \( [a, b] \).

3) \( z(t) \) is one-to-one on \( [a, b] \).

If all the above conditions hold except \( z(a) = z(b) \), we say \( \gamma \) is a smooth closed curve.
80 Examples:

Smooth Arc

Smooth Closed Curve

Not Smooth

Not Smooth

Corners and self-crossings make something not-smooth. Crossings ruin one-to-one property and corners means $z'(t)$ is not continuous or zero there.

If there is one "admissible" parametrization there are many.

Example: Find a parametrization of a line segment, $\gamma$, connecting $z_1$ and $z_2 \in \mathbb{C}$.

$$z(t) = z_1 \left( \frac{b - t}{b - a} \right) + z_2 \left( \frac{t - a}{b - a} \right), \quad t \in [a, b]$$

Usually we let $a = 0$ and $b = 1$:

$$z(t) = z_1 (1 - t) + z_2 t, \quad \text{for } t \in [0, 1].$$
\[ z(t) = z_0 \left( 1 - t \right) \] changes which direction we're going.

Giving a curve a direction makes it directed.

Example: suppose \( \gamma \) is a portion of a circle, centered at \( z_0 \) with radius \( r > 0 \), between angles \( \theta_1 \) and \( \theta_2 \).

\[ \gamma(t) = z_0 + t e^{i \left( \theta_1 \frac{t-a}{b-a} + \theta_2 \frac{t-a}{b-a} \right)} \quad \text{for} \quad t \in [a,b]. \]

Lots of specific examples in the book.

**Definition 2:** A contour \( \gamma \) is either a single point \( z_0 \) or a finite sequence of directed
Smooth curves \( \gamma_1, \gamma_2, \ldots, \gamma_n \) such that the last point of each \( \gamma_k \) is the first point of \( \gamma_{k+1} \).

We say \( \Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n \) and \(-\Gamma\) goes the opposite direction.

A parametrization of a contour is just a piecewise smooth curve.

Example: Parametrize \( \Gamma \):

\[
\begin{align*}
\gamma_1(t) &= (1-t) + (1+2i)t \\
&= e^{i(\pi(1-t)+(-\frac{\pi}{2})t)} \\
&= 1 + e^{i(\frac{\pi}{2}-\pi t)} \\
&\quad t \in [0,1].
\end{align*}
\]

\[
\gamma_2(t) = 1+i + e^{i(\frac{\pi}{2}(1-t)+(-\frac{\pi}{2})t)} = 1+i + e^{i(\frac{\pi}{2}-\pi t)} \\
&\quad t \in [0,1].
\]

Now we glue them together

\[
\gamma(t) = \begin{cases} 
\gamma_1(2t) & t \in [0,\frac{1}{2}] \\
\gamma_2(2(t-\frac{1}{2})) & t \in [\frac{1}{2},1]. 
\end{cases}
\]

And we're done.