So \( \varphi(x, y) = \left( \frac{\pi - \pi}{2} \right) x + \frac{\pi + \pi}{2} \).

In general, the strategy is to look for a known harmonic function which is similarly constant on the boundary.

§ 3.5 Complex Powers

What does \( z^i \) mean? Well, what does \( x^i \) mean for \( x \in \mathbb{R} \)?

\[ x^i = e^{i \ln x} \text{ for real } x > 0. \]

So we figure we do the same sort of thing for \( z^i \):

\[ z^i = e^{i \log(z)} \]

But \( \log(z) \) is multiple-valued, so \( z^i \) is not uniquely defined.

**Definition 5.** If \( \alpha \) is a complex constant and \( z \neq 0 \), then we define \( z^\alpha \) by

\[ z^\alpha = e^{\alpha \log z} \]
If a function, \( f(z) \), is multiple-valued then \( F(z) \) is a branch of \( f(z) \) on a region \( R \) if \( F(z) \) is continuous on \( R \) and agrees with one of the values of \( f(z) \).

**Example:** If \( f(z) = \log(z) \), \( F(z) = \sum (z) \) is a branch of \( f(z) \).

If \( g(z) = \log(z^2 - 1) \), \( G(z) = \sum (z^2 - 1) \) is a branch of \( g(z) \).

If \( h(z) = z^n \), \( H(z) = e^{\alpha \sum (z)} \) is a branch of \( h(z) \).

For this class, all examples of branches involve logarithm.

**Example:** Find all values of \( (z^i)^z \).

\[
\log(z^i) = \log|z|i\left(\frac{\pi}{2} + 2\pi k\right) \quad k \in \mathbb{Z}
\]

So

\[
(z^i)^z = e^{i\log|z|i\left(\frac{\pi}{2} + 2\pi k\right)} = e^{\frac{\pi i}{2}} e^{i \log|z|} e^{2\pi i k}
\]
\( z^\alpha = e^{(\log|z| + i\arg(z))} e^{2\pi i k} \)  \( k \in \mathbb{Z} \)

All roots lie on the \( \arg(z) = \log|z| \) line, and have varying magnitudes.

We see \( z^\alpha = e^{(\log|z| + i\arg(z))} e^{2\pi i k} \) \( k \in \mathbb{Z} \)

This set is finite only if different \( k \)'s give the same value, that is

\[ e^{2\pi i k_1} = e^{2\pi i k_2} \Rightarrow 2\pi i k_1 = 2\pi i k_2 + 2\pi i m \]

For some \( m \in \mathbb{Z} \), \( k_1 = k_2 \)

so \( \alpha k_1 = \alpha k_2 + m \Rightarrow \alpha = \frac{m}{k_1 - k_2} \) so \( \alpha \) is real +

This argument is reversible so

\( z^\alpha \) is finite iff \( \alpha \) is rational, \( \frac{m}{n} \)

Also \( z^\alpha \) is single-valued iff \( e^{2\pi i k} = 1 \) for all \( k \). So \( \alpha 2\pi i k = 2\pi i m \) so \( \alpha k = m \) so \( \alpha \in \mathbb{Z} \) for all \( k \in \mathbb{Z} \), so \( \alpha \in \mathbb{Z} \).
$e^z$ is single-valued iff $x \in \mathbb{Z}$.

We call $e^{x \log z}$ the principal branch of $z^x$ and it's analytic away from the branch cut, by chain rule:

$$\frac{d}{dz} z^x = \frac{d}{dz} e^{x \log z} = \frac{x}{z} e^{x \log z} = \frac{x}{z} z^x$$

Sort of like our power rule since $\frac{x}{z} z^x =^\sim x z^{x-1}$ provided we use the same branch cuts for $z^x$ and $z^{x-1}$.

Finding a branch of $(f(z))^x$ can be tedious and counter-intuitive.

Example: Find a branch of $(z^2-1)^{1/2}$ that is analytic for $|z| > 1$.

Let $g(z) = z^2 - 1$. Notice $g(z)$ takes $|z| > 1$ to $|z+1| > 1$. 

\[
\begin{array}{c}
\begin{array}{c}
\text{y} \\
\text{x} \\
\end{array}
\end{array}
\longrightarrow
\begin{array}{c}
\begin{array}{c}
\text{y} \\
\text{x} \\
\end{array}
\end{array}
\]
Every possible argument is hit by this domain, so $e^{\frac{i}{2} \pi} \sqrt{z^2 - 1}$ won't work for any $\pi i$. What do we do?

Let $(z^2 - 1)^{\frac{1}{2}} = z(1 - \frac{1}{z})^{\frac{1}{2}}

Both work since squaring both gives the desired answer. Let $h(z) = 1 - \frac{1}{z^2}$.

So any $z \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ will work, so our branch can be $z e^{\frac{i}{2} \pi} \sqrt{1 - \frac{1}{z^2}}$.

There's no general technique here, just be clever.