We can also define \( \tan z, \cot z, \sec z, \csc z \)

in terms of \( \sin z \) and \( \cos z \) in the usual

way and the derivatives are what you'd expect.

(i.e. \( \frac{d}{dz} \tan z = \sec^2 z \)).

Note however that \( |\sin z| \leq 1 \) when \( z \in \mathbb{R} \).

The range of \( \sin z \) and \( \cos z \) is much larger.

Indeed \( |\cos(iz)| = \left| \frac{e^x + e^{-x}}{2} \right| \\
\geq 1 \).

Also have hyperbolic trig functions:

**Definition:** \( \sinh z := \frac{e^z - e^{-z}}{2} = \frac{1}{i} \sin(iz) \)

\( z \in \mathbb{C} \):

\( \cosh z := \frac{e^z + e^{-z}}{2} = \cos(iz) \).

**Exercise:** \( \frac{d}{dz} \sinh z = \cosh z \), \( \frac{d}{dz} \cosh z = \sinh z \).

\( \tanh(z) := \frac{\sinh z}{\cosh z} \), similarly for \( \coth(z), \text{sech}(z), \csc(h)(z) \).

§ 3.3 The logarithmic function

We want to define \( \log z \) the same way we define it for real numbers:

\[ w = \log z \text{ if } z = e^w \]

The problem is that \( f(z) = e^z \) is not a one-to-one function, so
(15) \( \log z \) WHICH IS THE INVERSE OF \( e^z \) IS MULTIPLE-VALUED.

Let \( \text{Ln}(x) \) BE THE REAL, BASE \( e \) LOGARITHM YOU ARE FAMILIAR WITH:
\[
y = \text{Ln}(x) \iff e^y = x.
\]

This is well-defined for \( x > 0 \). So let \( z = re^{i\theta} \). Then we see \( z = e^{\text{Ln}(r)} e^{i\theta} \)
So \( z = e^{\text{Ln}(r)+i\theta} \), but also \( z = e^{\text{Ln}(r)+i(\theta+2\pi k)} \) \( k \in \mathbb{Z} \)
So we have \( \log z = \text{Ln}(r)+i\theta+2\pi ik \) \( k \in \mathbb{Z} \).

With \( z \neq 0 \), this is multiple-valued. Since \( r = |z| \)
and \( \theta+2\pi k = \text{Arg} z \) we see

**Definition 3:** If \( z \neq 0 \), then we define
\( \log z \) TO BE THE SET OF INFINITELY MANY VALUES:
\[
\log z = \text{Ln}|z|+i\text{Arg} z
\]
or \( \log z = \text{Ln}|z|+i\text{Arg} z + 2\pi ik, \ k \in \mathbb{Z} \).

**Example:** Find \( \log(2-2i) \).
We have \( |2-2i| = \sqrt{2^2+2^2} = 2\sqrt{2} \) AND \( \text{Arg}(2-2i) = -\frac{\pi}{4} \)
So \( \log(2-2i) = \text{Ln}(2\sqrt{2}) + i(2\pi k - \frac{\pi}{4}) \).

Also, since \( \text{Arg} z_1 z_2 = \text{Arg} z_1 + \text{Arg} z_2 \)
and \( \text{Arg} (\frac{z_1}{z_2}) = \text{Arg} z_1 - \text{Arg} z_2 \) we have
\[ \log z_1 z_2 = \log z_1 + \log z_2 \]

And \[ \log \frac{z_1}{z_2} = \log z_1 - \log z_2 \]

But these are multiple-valued (lower case).

Often, however, we want a single-valued function, so we can compute derivatives and such. Addition and division rules don't work for this.

Definition \[ \log z := \ln |z| + i \arg z \]

Call this the principal branch of logarithm. It is defined everywhere except 0, but it is also not continuous at real, negative \( z \), like \( \arg(z) \).

Let \( D^* \) be the entire complex domain except for the branch cut:

\[ y \]
\[ \text{everywhere but here} \]
\[ = D^* \]

Theorem 4: \( \log z \) is analytic on \( D^* \) with derivative \( \frac{1}{z} \).
\textbf{Proof:} \[ \lim_{\Delta z \to 0} \frac{\log(z_0 + \Delta z) - \log(z_0)}{\Delta z} \]

\[ = \lim_{\Delta z \to 0} \frac{\log(z_0 + \Delta z) - \log(z_0)}{\log(z_0 + \Delta z) - \log(z_0)} \quad \text{Let } W = \log(z_0 + \Delta z) \]

\[ W_0 = \log(z_0) \]

\[ = \lim_{W \to W_0} \frac{e^W - e^{W_0}}{W - W_0} = \frac{1}{e^{W_0}} = e^{W_0} = \frac{1}{z_0}. \]

This substitution works provided we are away from the branch cut.

We can define other branches of \( \log \) with other branches of \( \log \):

\[ S_c(z) = \ln |z| + i \arg_c(z), \text{ for } z \in \mathbb{R}. \]

We can also show \[ \frac{\partial}{\partial z} S_{c}(z) = \frac{1}{z} \text{ everywhere except its branch cut!} \]

So \textbf{Corollary 1:} \( \arg z \) is harmonic on \( \mathbb{D}^* \) and \( \arg \) is harmonic away from its branch cut.
\textbf{Corollary 2:} \( \ln |z| = \log |z| \) is harmonic everywhere but 0.

Both of these facts can be checked explicitly.

\textbf{Example:} Determine where \( \log (iz + 2) \) is analytic.

\( \log (z) \) is analytic except where \( z \) is real non-positive so \( \log (iz + 2) \) is analytic where \( iz + 2 \) is not real non-positive:

\[ iz + 2 = t \text{ where } t \in \mathbb{R}, \ t \geq 0 \]

so \( z = \frac{t - 2}{i} = i(2 - t) \) is where \( \log (iz + 2) \) is not analytic:

We can use the other branches of \( \log \) to deal with points where \( \log \) is undefined.
**Definition 4:** Let \( f(z) \) be a multiple-valued function (like \( \log(z) \)). We say \( F(z) \) is a branch of \( f(z) \) on a domain \( D \) if it is single-valued and continuous on \( D \) and \( F(z) \) is one of the values of \( f(z) \).

So like \( \log(z) \) and \( \log(z) \).

**Example:** Determine a branch, \( F(z) \), of \( F(z) = \log((z-3)^3) \) such that it is analytic at \( z = 1 \) and \( F(1) = \ln 18 + 3\pi i \).

\( F' \) for \( F(1) \).

Well, \( (1-3)^3 = -8 \) which is on the branch cut for \( \log z \) so we need a different branch of \( \log z \). Since \( \arg_{2\pi}(-8) = 3\pi \), we use \( \zeta_{2\pi}((z-3)^3) = F(z) \) since \( \zeta_{2\pi}(-8) = \ln 18 + 3\pi i \).

Also \( \frac{d}{dz} \zeta_{2\pi}((z-3)^3) = \frac{1}{(z-3)^3} \cdot 3(z-3)^2 \left|_{z=1}^{z=1} \right. = \frac{3}{z-3} \left|_{z=1}^{z=1} \right. = \frac{3}{1-3} = -\frac{3}{2} = F'(1) \).