Ultimately, these techniques will become valuable to us as we begin to compute contour integrals.

§ 3.2 THE EXPONENTIAL, TRIGONOMETRIC, AND HYPERBOLIC FUNCTIONS

Recall $e^z = e^{x+iy} = e^x e^{iy} = e^x \cos(y) + i \sin(y)$, $x, y \in \mathbb{R}$.

\[ \frac{d}{dz} e^z \text{ AND } |e^z| = e^x \text{ since } |e^{iy}| = 1 \text{ for } y \in \mathbb{R}. \]

\[ \arg e^{x+iy} = y + 2\pi k, \text{ } k \in \mathbb{Z}. \]

A function $f(z)$ is one-to-one if $f(z_1) = f(z_2)$ means $z_1 = z_2$. We see $e^z$ is not one-to-one.

Indeed:

**Theorem 3:**

1. $e^z = 1 \iff z = 2\pi i k, \text{ } k \in \mathbb{Z}$
2. $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2\pi i k, \text{ } k \in \mathbb{Z}$

Why? a) $e^{x+iy} = 1 \iff e^x e^{iy} = 1$

Since $\arg e^{iy} = \arg 1$ we have $y = 2\pi k$. Since $|e^x| = |1| = 1$, $x = 0$ so $z = 0 + 2\pi i k$.

b) $e^{z_1} = e^{z_2} \iff e^{z_1 - z_2} = 1 \iff z_1 - z_2 = 2\pi i k$
WE SAY \( f(z) \) IS **PERIODIC** IF THERE EXISTS \( \lambda \in \mathbb{C} \) SUCH THAT \( f(z + \lambda) = f(z) \) FOR ALL \( z \in \mathbb{C} \).

WE CALL \( \lambda \) A **PERIOD** OF \( f(z) \).

So \( 2\pi i \) is a period of \( f(z) = e^z \).

**WHAT DOES THIS MEAN?**

Each one of these horizontal strips

\[ S_n := \frac{\pi}{2} x + iy \mid -\infty < x < \infty, \quad (2n-1)\pi < y \leq (2n+1)\pi \quad n = 0, \pm 1, \pm 2, \ldots \]

maps to all \( C \) (except 0) via the \( e^z \) map.

Indeed, AND horizontal strip of width \( 2\pi \) WOULD DO. CALL THIS STRIPS THE FUNDAMENTAL REGION FOR \( e^z \).

Now for some complex trig functions!

For \( y \in \mathbb{R} \) we have \( \cos(y) = \frac{e^{iy} + e^{-iy}}{2} \), \( \sin(y) = \frac{e^{iy} - e^{-iy}}{2i} \).
So we can do the same for $z \in \mathbb{C}$

**Definition 1.** Given any $z \in \mathbb{C}$, we define

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}; \quad \cos z := \frac{e^{iz} + e^{-iz}}{2}$$

So now we have $\sin z$ and $\cos z$ for complex angles.

**Note**

$$\frac{d}{dz} \sin(z) = \frac{d}{dz} \frac{e^{iz} - e^{-iz}}{2i} = \frac{i e^{iz} + i e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2}$$

$$= \cos(z)$$

$$\frac{d}{dz} \cos(z) = \frac{d}{dz} \frac{e^{iz} + e^{-iz}}{2} = \frac{-i e^{iz} - i e^{-iz}}{2i} = -\frac{e^{iz} + e^{-iz}}{2i}$$

$$= -\sin(z)$$

So that still works.

Indeed, all standard trig identities still hold.

(See page 113).

**Example:** Show $\sin^2 z + \cos^2 z = 1$

$$\left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz})$$

$$- \frac{1}{4}(e^{2iz} - 2 + e^{-2iz})$$

$$= \frac{2}{4} + \frac{2}{4} = 1.$$
We can also define \( \tan z, \cot z, \sec z, \csc z \)

in terms of \( \sin z \) and \( \cos z \) in the usual way and the derivatives are what you'd expect.

(i.e. \( \frac{d}{dz} \tan z = \sec^2 z \)).

Note however though \( |\sin z| \leq 1 \) when \( z \in \mathbb{R} \).

The range of \( \sin z \) and \( \cos z \) is much larger.

Indeed \( |\cos(iy)| = \left| \frac{e^z + e^{-z}}{2} \right| = 1 \).

Also have hyperbolic trig functions:

**Definition 2:** \( \sinh z := \frac{e^z - e^{-z}}{2} = \frac{1}{i} \sin(i z) \)

\( z \in \mathbb{C} \) : \( \cosh z := \frac{e^z + e^{-z}}{2} = \cos(i z) \).

Exercise

\[ \frac{d}{dz} \sinh z = \cosh z, \quad \frac{d}{dz} \cosh z = \sinh z. \]

\( \tanh(z) := \frac{\sinh z}{\cosh z} \), similarly for \( \coth(z), \text{sech}(z), \text{csch}(z) \).

\[ \S 3.3 \text{ THE LOGARITHMIC FUNCTION} \]

We want to define \( \log z \) the same way we define it for real numbers:

\( w = \log z \) if \( z = e^w \)

The problem is that \( f(z) = e^z \) is not a one-to-one function, so
\[ \log z \] which is the inverse of \( e^z \) is \textit{multiple-valued}.

Let \( \ln(x) \) be the real, base-2 logarithm you are familiar with:
\[ y = \ln(x) \iff e^y = x. \]

This is well-defined for \( x > 0 \). So let \( z = re^{i\theta} \). Then we see \( z = e^{\ln(r)e^{i\theta}} \).

So \( z = e^{\ln(r)+i\theta} \), but also \( z = e^{\ln(r)+2\pi i k} \) \( \forall k \in \mathbb{Z} \).

So we have \( \log z = \ln(r)+i\theta+2\pi i k \) \( \forall k \in \mathbb{Z} \).

With \( r > 0 \) and \( \theta+2\pi k = \arg z \) we see \( \boxed{\text{Definition 3: If } z \neq 0, \text{ then we define } \log z \text{ to be the set of infinitely many values:}} \)

\[ \log z = \ln |z| + i \arg z \]

or \( \log z = \ln |z| + i \text{Arg} \, z + 2\pi i k \), \( k \in \mathbb{Z} \).

\[ \underline{\text{Example: Find } \log(2-2i).} \]

Well \( |2-2i| = \sqrt{2^2+2^2} = 2\sqrt{2} \) and \( \text{Arg} \, (2-2i) = \frac{-\pi}{4} \).

So \( \log(2-2i) = \ln(2\sqrt{2}) + i(2\pi k - \frac{\pi}{4}) \).

Also, since \( \text{arg } z \, \text{st} = \text{arg } z_1 + \text{arg } z_2 \)

and \( \text{arg} \left( \frac{z_1}{z_2} \right) = \text{arg } z_1 - \text{arg } z_2 \) we have