NICE THING ABOUT POLYNOMIALS, WE CAN
 FACTOR THEM. BY USING POLYNOMIAL LONG
 DIVISION, WE CAN SHOW THAT IF $P_n(z) = 0$
 THEN $z_1$ IS A ROOT OF $P_n(z)$ AND THAT
 $P_n(z) = (z - z_1)q_{n-1}(z)$ WHERE $q_{n-1}(z)$ IS A
 POLYNOMIAL OF DEGREE $n-1$. WE CAN MAYBE
 REPEAT THIS PROCESS. BOOK CALLS THIS
 "DECOMPOSING" $P_n$.

EXAMPLE: $P_3(z) = z^3 - (2+2i)z^2 - (3-4i)z + 6i$

WE SEE $P_3(-i) = -1 - 2 - 2i + 3 + 4i + i = 0$

SO $P_3(z) = (z + i)q_2(z)$. WE FACTOR $P_3(z)$
WITH POLYNOMIAL LONG-DIVISION

\[
\begin{align*}
\frac{z^2 - (3+2i)z + 6i}{(z + i)(z^3 - (2+2i)z^2 - (3-4i)z + 6i)} \\frac{z^3 + z^2}{z^3 + (3+2i)z^2 - (3-4i)z} \\frac{-(3+2i)z^2 - (3+2i)z}{-(3+2i)z^2 - (3+2i)z + 6i} \\frac{6i z + 6i}{6i z + 6i} \\frac{0}{0}
\end{align*}
\]

SO $P_3(z) = (z + i)(z^2 - (3+2i) + 6i)$

\[\text{LINEAR FACTOR (DEGREE 1)} \quad \text{QUADRATIC FACTOR (DEGREE 2)}\]
We can factor $z^2 - (3+2i) + (3-2i)$ using the quadratic equation:

$$z = \frac{3+2i \pm \sqrt{(3-2i)^2 - 24i}}{2}$$

$$z = \frac{3+2i \pm \sqrt{5-12i}}{2}$$

We know how to find square roots of complex numbers.

Exercise: Compute $\sqrt{5-12i}$.
Show it's $\pm (3 - 2i)$.

So $z = 2i, 3$.

$p_3(z) = (z+1)(z-2i)(z-3)$.

How do we know $p_n(z)$ has any zeros? Certainly $x^4 + 3x^2 + 2$ doesn't have any over the real numbers?

We can thank Gauss for that.

**Theorem 1.** (Fundamental Theorem of Algebra)

Every non-constant polynomial $a_n z^n + \cdots + a_0 z^0$, with $a_n \neq 0$, has at least one zero in $\mathbb{C}$. 

This gives us that a degree $n$ polynomial factors as a product of $n$ (possibly repeating) linear factors:

$$a_n x^n + \cdots + a_1 x + a_0 = a_n (x - z_1) (x - z_2) \cdots (x - z_n)$$

where $z_j \in \mathbb{C}$.

Why?
Because if $z_1$ is one zero of $p_n(z)$ then $p_n(z) = (z - z_1) q_{n-1}(z)$ and we can repeat the process inductively on $q_{n-1}(z)$.

So if two polynomials have the same set of (possibly repeating) roots then they are constant multiples of each other.

**Example:** Suppose $p_n(x) = a_n x^n + \cdots + a_1 x + a_0$ where one or and $x$ is a real variable. Show that $p_n(x)$ factors as a product of linear and quadratic factors over the real numbers.

A real polynomial can be considered a complex polynomial: $p_n(z) = a_n z^n + \cdots + a_1 z + a_0$ now $p_n(x)$ factors over $\mathbb{C}$ so $p_n(x) = a_n (x - z_1) \cdots (x - z_n)$ if $z_1$ is a root of $p_n(z)$ then
\[ p_n(z) = a_n z^n + \ldots + a_1 z + a_0 = 0 \]

So
\[ \overline{p_n(z)} = \overline{a_n z^n + \ldots + a_1 z + a_0} = a_n \overline{z}^n + \ldots + a_1 \overline{z} + a_0 = \overline{p_n(\overline{z})} = 0 \]

\[ \overline{a_j} = a_j \] since each \( a_j \) is real.

So \( \overline{z} \) is also a root of \( p_n(z) \). So either \( z_1 = \overline{z} \) and \( z \) is real or else \( z_1 \) has a conjugate pair as a root.

So
\[ p_n(z) = (z - z_1)(z - \overline{z_1})\ldots(z - z_k)(z - \overline{z_k})(z - z_m)\ldots(z - z_n) \]

Real linear factors

But \( (z - z_1)(z - \overline{z_1}) = z^2 - (z + \overline{z})z + |z|^2 \overline{\rho(z_1)} \) \( \overline{\rho(z_1)} \) \( \text{real} \)

So these are quadratic polynomials with real coeff. So we are done.

Eventually we will want to do Taylor series with complex functions, polynomials. Let us practice those.

Example: Express \( p_3(z) = z^3 - (2 + 1i)z^2 - (3 - 4i)z + 6i \) as a polynomial in terms of powers of \( (z - 2) \) instead of \( z \).
So \( p_3(z) = a_3(z-2)^3 + a_2(z-2)^2 + a_1(z-2) + a_0 \)

For some \( a_i \in \mathbb{C} \).

Observe \( p_3(z) = a_0 \) because the \((z-2)^k\)

Factors vanish. Since \( p_3(z) = (z+1)(z-2i)(z-3) \)

\[ p_3(z) = (3)(2-2i)(-1) = -6 + 6i = a_0 \]

Now note that \( p_3'(z) = 3a_3(z-2)^2 + 2a_2(z-2) + a_1 \)

So \( p_3'(z) = a_1 \) and

\[ p_3''(z) = 6a_3(z-2) + 2a_2 \]

So \( a_2 = \frac{p_3''(z)}{2} \)

\[ p_3'''(z) = 6a_3 \] So \( \frac{p_3'''(z)}{6} = a_3 \)

Solving all of these we get

\[ a_1 = p_3'(z) = 1 - 4i \]

\[ a_2 = \frac{p_3''(z)}{2} = 4 - 2i \]

\[ a_3 = \frac{p_3'''(z)}{6} = 1 \]

So \( p_3(z) = (z-2)^3 + (4-2i)(z-2)^2 + (1-4i)(z-2) - 6 + 6i \)

We see we can generalize this reasoning to get that we can take any polynomial of degree \( n \), \( p_n(z) \), and make it centered at \( z_0 \) by showing

\[ p_n(z) = \frac{p_n^{(0)}(z_0)}{0!}(z-z_0)^0 + \frac{p_n^{(1)}(z_0)}{1!}(z-z_0)^1 + \cdots + \frac{p_n^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \frac{p_n^{(n)}(z_0)}{n!}(z-z_0)^n \]
\[ P_n(z) = \sum_{k=0}^{n} \frac{p_k(z_0)}{n!} (z-z_0)^k \quad \text{where } p_k \text{ means } k^{th} \text{ derivative.} \]

This is the Taylor form of \( P_n(z) \) at \( z_0 \).

When \( z_0 = 0 \), this is the standard form of \( P_n(z) \), or Maclaurin form.

Now back to rational functions. Let

\[ R_{m,n}(z) = \frac{a_m z^m + \ldots + a_0}{b_n z^n + \ldots + b_0} = \frac{a_m (z-z_1) \ldots (z-z_m)}{b_n (z-w_1) \ldots (z-w_n)} \]

Since the polynomials factor, we call the \( z_i \) the zeros of \( R_{m,n}(z) \) and the \( w_j \) are the poles of \( R_{m,n}(z) \). We see \( |R_{m,n}(z)| \to 0 \) as \( z \to w_j \).

We say the number of times the same \( z_i \) or \( w_j \) appears (after simplifying) is the multiplicity of the zero or pole.

Example: Find the zeros, poles and multiplicities of \( f(z) = \frac{(z-2i)^5(z+2)}{(z^2+4)^8} z^2 \) simplifying.

We see \( f(z) = \frac{2(z-2i)^5(z+1)}{(z+2i)^8(z-2i)^8} z^2 = \frac{z(z+1)}{(z+2i)^8(z-2i)^3} z^2 \)

So zero at \( z = -1 \) with multiplicity 1.
38. POLES AT \( z = -2i \) WITH MULT. 8
   \( z = 2i \) WITH MULT. 3
   \( z = 0 \) WITH MULT. 2.

REMEMBER PARTIAL FRACTION EXPANSIONS?

\[
\frac{3z^2 + 4z - 5}{(z-2)(z+1)(z+3)} = \frac{1}{z-2} + \frac{1}{z+1} + \frac{1}{z+3}
\]

\[
\frac{4z + 4}{z(z-1)(z-2)} = -\frac{1}{z} + \frac{8}{z-1} + \frac{7}{z-2} + \frac{6}{(z-2)^2}
\]

BEING ABLE TO REDUCE ANY POLYNOMIAL IN \( \mathbb{C} \) AS A PRODUCT OF LINEAR FACTORS ALLOWS US TO SIMILARLY EXPAND ANY RATIONAL FUNCTIONS.

**THEOREM 2.3** If

\[
\mathcal{R}_{m,n}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m}{b_m (z-z_1)^{d_1}(z-z_2)^{d_2} \ldots (z-z_r)^{d_r}}
\]

IS A RATIONAL FUNCTION WHOSE DENOMINATOR DEGREE \( n = d_1 + d_2 + \ldots + d_r \) \( > \) \( m \) THEN \( \mathcal{R}_{m,n}(z) \) HAS A PARTIAL FRACTION DECOMPOSITION OF