We say \( z_0 \) is a singularity of \( f(z) \) if \( f(z) \) is analytic on a neighborhood of \( z_0 \) except at \( z_0 \) itself.

In the previous example, \( z = -1 \) was a singularity.

**Example:** \( f(z) = \text{Re}(z) \) is nowhere diff.

\( f(z) = \frac{z + \overline{z}}{2} = \frac{z}{2} + \frac{\overline{z}}{2} \) so \( 2f(z) - z = \overline{z} \). Since \( z \) is entire, if \( f(z) \) is diff. at \( z_0 \) then \( 2f(z) - z = \overline{z} \) is diff. at \( z_0 \). But \( \overline{z} \) is nowhere diff., so \( f(z) \) is nowhere diff.

But there are easier ways to determine if \( f(z) \) is diff.

§ 2.4 \textbf{The Cauchy-Riemann Equations}

We would like to tell if \( f(z) = u(x,y) + i v(x,y) \) is diff. just by looking at \( u \) and \( v \).

Recall that \( f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \)

Since \( \Delta z = \Delta x + i \Delta y \) we can let \( \Delta z \) go to zero by fixing \( \Delta y = 0 \) and letting \( \Delta x \to 0 \) so
\[ \text{IF LIMIT EXISTS,} \]
\[ \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{f(x_0 + i \Delta x + i \Delta y) - f(x_0, y_0)}{\Delta z} \]
\[ = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - (u(x_0, y_0) + iv(x_0, y_0)) + i}{\Delta x} \]
\[ = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \]
\[ = \frac{\partial u}{\partial x} (z_0) + i \frac{\partial v}{\partial x} (z_0) = f'(z_0) \]

**But we could’ve also fixed \( \Delta x = 0 \) and let \( \Delta y \to 0 \)**

\[ \lim_{\Delta y \to 0} \frac{f(x_0 + i(y_0 + \Delta y)) - f(x_0, y_0)}{i \Delta y} = \frac{f(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{i \Delta y} \]
\[ = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i \Delta y} + \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y} \]
\[ = \frac{\partial v}{\partial y} (z_0) - i \frac{\partial u}{\partial y} (z_0) = \overline{f'(z)} \]
\[ = \frac{\partial u}{\partial x} (z_0) + i \frac{\partial v}{\partial x} (z_0) \]

**These limits must agree so**
We have the Cauchy-Riemann equations
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \] (CREs)

must hold at \( z_0 \) if \( f(z) \) is diff. at \( z_0 \).

Theorem 4. A necessary condition for a function \( f(z) = u(x,y) + iv(x,y) \) to be diff. at a point \( z_0 \) is that the CREs hold at \( z_0 \). Furthermore, if \( f(z) \) is analytic on an open set \( G \), then the CREs hold at every point in \( G \).

How do I remember CREs?

Remember that \( \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \)

\[ \Rightarrow \frac{\partial u}{\partial x} = \frac{1}{i} \frac{\partial (u+iv)}{\partial y} \]

\[ \Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \]

Compare real + imaginary parts

Example: \( f(x,y) = x^2 - y^2 - 2ixy \) is not analytic anywhere.

\( u(x,y) = x^2 - y^2 \Rightarrow \frac{\partial u}{\partial x} = 2x \) and \( \frac{\partial u}{\partial y} = -2y \)

\( v(x,y) = -2xy \Rightarrow \frac{\partial v}{\partial x} = -2y \) and \( \frac{\partial v}{\partial y} = -2x \)

If CREs hold \( 2x = -2x \) and \( -2y = 2y \)
**This is only possible if** \( x = 0 \) and \( y = 0 \), so the set of points where \( f(z) \) is potentially analytic is not an open set.

We have a stronger theorem.

**Theorem 5.** Let \( f(z) = u(x, y) + iv(x, y) \) be defined on some open set \( G \) containing \( z_0 \), if the first partial derivatives \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \) exist and are continuous in \( G \) at \( z_0 \), then \( f(z) \) is diff. at \( z_0 \).

Consequently, if CREs hold on all of \( G \) then \( f(z) \) is analytic on \( G \).

**Proof:** See book. This is not a proof-intensive class, uses mean value theorem for real #s.

So \( f(x) = x^2 - y^2 - 2ixy \) is indeed diff. at \( z_0 = 0 \).

In previous example.

**Example:** Prove \( f(z) = e^z \) is entire and find its derivative.
\[ f(z) = e^z e^{i\gamma} = e^x (\cos y + i \sin y) \]

\[ u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y \]

so \[ \frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \] and \[ \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x} \]

so \text{CRES ARE SATISFIED EVERYWHERE AND PARTIAL DERIVS ARE CONTINUOUS EVERYWHERE. SO } f(z) \text{ IS ENTIRE. FURTHERMORE} \]
\[ \frac{d}{dz} f(z) = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \]
\[ = e^x \cos y + i e^x \sin y \]
\[ = e^z. \]

so \[ \frac{d}{dz} e^z = e^z. \]

**Theorem 6.1** If \( f(z) \) is analytic on the domain \( D \) (open, connected) and if \( f'(z) \neq 0 \) everywhere then \( f(z) \) is constant on \( D \).

Why? \[ f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} - i \frac{\partial u}{\partial x} = 0 \]
on \( D \), so \[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \] and \[ \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0. \]

if the partials of \( u \) and \( v \) are 0 on a domain, they are constant on \( D \). so \( f = u + iv \) is constant on \( D \).