MTHE 228: COMPLEX ANALYSIS

DR. THOMAS A. HULSE ("TOM")

hulse@mast.querasu.ca

READ THE SYLLABUS: www.mast.querasu.ca/~hulse/228.htm

WHAT'S SO GREAT ABOUT COMPLEX NUMBERS?

LOTS OF THINGS

- POLYNOMIALS HAVE ROOTS

- ELEGANT GEOMETRIC INTERPRETATION

- CONCEPT OF DIFFERENTIABILITY IS STRONGER

- CAN MAKE SOME KINDS OF INTEGRATION EASIER

- CLOSE RELATIONSHIP WITH HARMONIC FUNCTIONS
  (RELATES TO THINGS LIKE HEAT AND ELECTRICITY+ MAGNETISM)

- FOURIER ANALYSIS, INTEGRAL TRANSFORMS, ETC.
  (USEFUL TOOLS FOR SCIENTISTS AND ENGINEERS)

SO LET'S START AT THE VERY BEGINNING

CHAPTER 1: COMPLEX NUMBERS

2.1.1 THE ALGEBRA OF COMPLEX NUMBERS

WHAT DOES √2 MEAN?

A SOLUTION TO \( x^2 = 2 \).

SO WHAT DOES √(-1) MEAN?

A SOLUTION TO \( x^2 = -1 \) \( \Rightarrow \) \( √(-1)^2 = -1 \)
2. We denote \( i = \sqrt{-1} \) (engineers sometimes use \( j \)). Call \( i \) "imaginary" because it can't be closely approximated by rational numbers (such numbers are called "real"). The word "imaginary" is a bit of a misnomer; they are no more or less existant than "real" numbers, they just encode information differently than the real numbers.

Let \( \mathbb{R} \) := real numbers, \( x \in \mathbb{R} \iff \text{"} x \text{ is a real number."} 

**Definition 1:** A complex number \( z \) is an expression of the form \( x + iy \) where \( x, y \in \mathbb{R} \). We say \( z \in \mathbb{C} := \text{complex numbers} \) and that if \( w = \alpha + i \beta \) with \( \alpha, \beta \in \mathbb{R} \) that \( z = w \iff (x = \alpha \text{ and } y = \beta) \).

Generally, if I write \( z = x + iy \) (or \( \alpha + i \beta \)) you can assume \( x, y \in \mathbb{R} \) (\( \alpha, \beta \in \mathbb{R} \)) unless I say otherwise.

In such a case

**Definition 2:** \( x = \text{Re}(z) \) is the real part of \( z \)
\( y = \text{Im}(z) \) is the imaginary part of \( z \)

If \( \text{Re}(z) = 0 \) we say \( z \) is a (pure) imaginary number.

Operations work like you would expect

**Addition**

**Subtraction**! \( (a + ib) + (x + iy) = (a + x) + i(b + y) \)

Like the addition of vectors \( (a, b) + (x, y) = (a + x, b + y) \).
**Multiplication:**

\[(a+ib)(x+iy) = ax + ibx + aiy + ibiy\]

\[= ax + ibx + iay + i^2 by\text{, with }i^2 = -1\]

\[= (ax - by) + i(ay + bx)\text{, real numbers}\]

\[= (x+iy)(a+ib)\text{, multiplication in }\mathbb{C}\text{, is commutative}\]

**Division:** If \(a+ib \neq 0+0i\) then

\[
\frac{x+iy}{a+ib} = \frac{(x+iy)(a-ib)}{(a+ib)(a-ib)} = \frac{(ax+by) + i(ay-bx)}{a^2 + b^2} = \frac{ax+by}{a^2 + b^2} + i\frac{ay-bx}{a^2 + b^2}.
\]

"Rationalize the denominator""

It is easy to check that these operations satisfy the following rules (same as rational and real #s) for \(\mathbb{C}, \mathbb{W}, \mathbb{U}, \text{ etc.}\):

**Commutative Laws of Addition and Multiplication:**

\[\mathbb{C} + \mathbb{W} = \mathbb{W} + \mathbb{C}, \quad \mathbb{C} \mathbb{W} = \mathbb{W} \mathbb{C}\]

**Associative Laws of Addition and Multiplication**

\[\mathbb{C} + (\mathbb{W} + \mathbb{U}) = (\mathbb{C} + \mathbb{W}) + \mathbb{U}\]

\[\mathbb{C}(\mathbb{W} \mathbb{U}) = (\mathbb{C} \mathbb{W}) \mathbb{U}\]

**Distributive Law:**

\[(\mathbb{C} + \mathbb{W}) \mathbb{U} = \mathbb{C} \mathbb{U} + \mathbb{W} \mathbb{U}\]

\(\mathbb{C}\) satisfies field axioms (in case you care).