1. (Section 4.6) Let $D$ be the domain inside the square with corners $0, 1, i,$ and $1 + i,$ not including the boundary. Show that $|z^2 - 1| < \sqrt{5}$ for all points in $D.$ Is there a better bound? [Hint: Try parametrizing the boundary with some $\gamma(t)$ and use calculus of real numbers to maximize $|\gamma(t)^2 - 1|.$]

2. (Section 4.6) Suppose $f(z)$ is entire and $|f(z)| < M|z|$ for some $M > 0$ for all $z \in \mathbb{C}.$ Show that $f(z) = Cz + D$ for some $C, D \in \mathbb{C}.$ [Hint: Use Theorem 20 in Section 4.6 to show $f'(z)$ is bounded, then use another theorem about bounded entire functions.]

3. (Section 4.7) Find all harmonic $\phi(x, y)$ on the open disk $|z - i| < 2$ with $|\phi(x, y)| \leq \pi$ and $\phi(0, 0) = -\pi.$

4. (Section 4.7) Suppose $\phi(x, y)$ is harmonic on the open disk $|z| < 5$ and on the boundary has the values:

$$\phi(5e^{i\theta}) = \begin{cases} 3 & \text{for } \theta \in (0, \pi) \\ 1 & \text{for } \theta \in (\pi, \frac{3\pi}{2}) \\ 2 & \text{for } \theta \in (\frac{3\pi}{2}, 2\pi) \end{cases}$$

a) Set up, but do not evaluate, the integral(s) for the value of $\phi(1, 1).$ Your answer should have no unspecified constants or variables except for the variable of integration. [Hint: Your answer will probably have three integrals in it, that’s fine.]

b) Find $\phi(0, 0).$

5. (Section 4.7) Let $D$ be all of $\mathbb{C}$ except for $z = 0.$ We know that $\text{Log}|z|$ is harmonic on $D$ but that its harmonic conjugate must be $\text{arg}_r(z),$ which is never harmonic on $D$ because of the branch cut. Explain why this doesn’t contradict Theorem 25 in section 4.7 of the book.

6. (Section 5.1) Evaluate the following series:

a) $\sum_{n=1}^{\infty} \frac{3i}{(2 + i)^{2n}}$

b) $\sum_{k=1}^{\infty} i^k \left(\frac{2}{3 + 3i}\right)^{k-1}$

c) $\sum_{\Theta=0}^{\infty} (3i)^{1-\Theta}$
7. (Section 5.1) Use whatever tests you like to determine if the sums converge or diverge. Make sure to explain your reasoning. Assume all exponents use the principal branch of log.

a) \[ \sum_{n=1}^{\infty} \frac{2}{(in)^{3+i}} \]

b) \[ \sum_{n=1}^{\infty} \frac{\sin(\pi n + i)}{n^2} \]

c) \[ \sum_{k=2}^{\infty} \frac{k^5}{(-2 + 5i)^k} \]

Note: Remember the p-series test from calculus which says, for \( p \in \mathbb{R} \),

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \]

converges if and only if \( p > 1 \).