INSTRUCTIONS: The exam has six questions labelled 1 through 6. Each question is worth 10 points.

The exam is two hours in length.

To receive full credit you must explain your answers. Partial credit is awarded for progress towards a correct answer. You do not need to re-prove things already proven in class or were asked to be proven in collected homework.

Write all answers on the exam. You may use the backs of pages if necessary.

Queen’s approved, non-programmable calculators are permitted. But all other notes, texts, data sheets and other aids are not permitted.

If you have a question about any of the problems, raise your hand and I will come to you. If you are having a hard time getting my attention, you can just call my name (Tom).

Good luck!

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1. — 
   a) Describe and sketch the set of points \( z \in \mathbb{C} \) where \( |z - (-1 - 2i)| \leq 3 \).

   b) Describe and sketch the image of the set in part (a) under the function,

   \[
   f(z) = \frac{1}{2i}(z - \overline{z}) + i - 1.
   \]
2. —
   a) In the partial fraction decomposition of
   \[ f(z) = \frac{iz^5 - (15 - 2i)z^4 - (24 + 90i)z^3 + (270 - 108i)z^2 + (216 + 405i)z - (238 - 162i)}{(z + 3i)^6}, \]
   find the coefficient of \(1/(z + 3i)\). That is, find the residue of \(f(z)\) at \(z = -3i\).

   b) Give the Taylor expansion of \(p(z) = 1 + z + z^2 + z^3\) at \(z = i\).
3. — Describe and sketch where

\[ f(z) = \begin{cases} 
3x(1 - i) + (y^2 + 3iy) & \text{for } |\text{Re } z| > 1 \\
x^2 + 2x + 2ixy + 2iy - y^2 & \text{for } |\text{Re } z| < 1 
\end{cases} \]

is complex differentiable. Where is it analytic?
4. —
a) Show that if $\psi(x, y)$ is a harmonic conjugate for $\phi(x, y)$ then $-\phi(x, y)$ is a harmonic conjugate for $\psi(x, y)$.

b) Suppose $f(z)$ is defined on $\mathbb{C}$ where

$$u(x, y) := \text{Re}(f(z)) = x^2 \cos(y) - y \cos(y)$$

Is it possible for $f(z)$ to be entire? If no, explain why. If yes, give $v(x, y) = \text{Im}(f(z))$ so that $f(z)$ is analytic.
5. — Let

\[ f(z) = \begin{cases} 
  z^2 & \text{for } 0 \leq \text{Arg}(z) \leq \frac{\pi}{4} \\
  z^3 & \text{for } \frac{\pi}{4} < \text{Arg}(z) \text{ or } \text{Arg}(z) < 0 \\
  0 & \text{for } z = 0
\end{cases} \]

Explain why \( f(z) \) is continuous at \( z = 0 \) but not continuous at \( z = 1 + i \). You do not need to make an \( \epsilon \) and \( \delta \) argument, but you can if you want to. In any case, you should probably draw a picture.
6. — Describe, and draw a picture of, where

\[ g(z) = \log( e^{\frac{-2\pi i}{3}z^2} + 1 ) \]

is analytic.