MTHE 228: Complex Analysis
Conceptual Solutions
Winter 2015

Find below a list of short conceptual questions that can always be answered with at most a couple of sentences and maybe an equation. None of these questions are computational, but they may have relevance to problems that you will see on the final exam. While there are a lot of questions on this list, it is not intended to be comprehensive. Rather, the preferred way to use this list of questions is to study the associated section first and then go through the questions for that section. These are not meant to be hard, so if there are questions you do not know how to answer it would probably be a good idea to go over that material more carefully.

Mathematics is about bringing ideas together to solve problems. Actual exam questions will probably not be this straight-forward, but they may require your understanding of these facts to answer a given question.

Chapter 1: Complex Numbers
Section 1.1: The Algebra of Complex Numbers

1) What is \( \text{Im}(z) \) and is it real or imaginary?

It is the imaginary part of \( z \), the real number multiplied by \( i \) in the representation of \( z \). It is real.

2) How do you write \( \frac{a+ib}{c-id} \), where \( a, b, c, d \in \mathbb{R} \) in the form \( x+iy \) where \( x, y \in \mathbb{R} \)?

You “rationalize the denominator” by multiplying the numerator and denominator by \( c-id \) and get \( \frac{ac-bd}{c^2+d^2} + i \frac{ad+bc}{c^2+d^2} \).

Section 1.2: Point Representation of Complex Numbers

3) How do we represent a complex number, \( z = a + ib \) for \( a, b \in \mathbb{R} \), on the plane?

As a point, \((a, b)\), where \( a \) is the \( x \)-coordinate and \( b \) is the \( y \)-coordinate.

4) What is the geometric interpretation of adding two complex numbers?

It is vector addition, we add the \( x \)-coordinate and the \( y \)-coordinates.

5) What is the complex conjugate and what is its geometric interpretation?

If \( z = a + ib \) for \( a, b \in \mathbb{R} \) then \( \overline{z} = a - ib \). On the complex plane it is a reflection over the real axis.

6) What is the modulus and what is its geometric interpretation?

If \( z = a + ib \) for \( a, b \in \mathbb{R} \) then \( |z| = \sqrt{a^2 + b^2} \). It is the distance between \( z \) and the origin.
7) How can we use the conjugate to compute the modulus?

It is just $|z|^2 = z\overline{z}$ so $|z| = \sqrt{z\overline{z}}$. 

8) Use the modulus to describe the distance between the points $\alpha$ and $\beta$ in $\mathbb{C}$.

It is just $|\alpha - \beta|$

9) Describe the set of points $z \in \mathbb{C}$ where $|z - 2i| = 7$.

It is a circle of radius 7 centered at $2i$.

10) Describe the set of points $z \in \mathbb{C}$ where $\text{Re}(z) = 4$.

It is the vertical line $x = 4$.

Section 1.3: Vectors and Polar Forms

11) What is the triangle inequality?

Just $|z + w| \leq |z| + |w|$. It comes from the fact that the sums of the lengths of two sides of a triangle are greater than the length of the remaining side.

12) What is the polar form of a complex number?

Since a point $a + ib$ on the complex plane can be given by an angle, $\theta$, relative to the positive $x$ direction and a distance, $r = |a + ib|$, from the origin, we have $a + ib = r \text{cis}(\theta)$.

13) Why does the argument of a complex number have infinitely many values?

If $z$ can be given in polar form with the angle $\theta$ then alternately the angle could be $\theta + 2\pi k$ where $k$ is any integer, which is an infinite set.

14) What is a branch of argument and what is its branch cut?

Since there are infinitely many possible values for argument, we specify a range of range of particular values, like $(-\pi, \pi]$ for $\text{Arg}(z)$, to make argument single valued. The branch cut occurs at arguments at the endpoints of this range and is where a branch of argument is discontinuous. Another branch would be $\text{arg}_\tau(z)$ which takes values in $(\tau, \tau + 2\pi]$.

15) What is the geometric interpretation of multiplying two complex numbers, $z$ and $w$?

You multiply the moduli (radii) of $z$ and $w$ and add their arguments.

16) What is the geometric interpretation of dividing two complex numbers?

You divide the moduli (radii) of $z$ and $w$ and subtract their arguments.
Section 1.4: The Complex Exponential

17) What is the definition of $e^z$ where $z = x + iy$ for $x, y \in \mathbb{R}$?

We have $e^{x+iy} = e^x \cos(y) + i \sin(y)$ where $e^x$, $\sin(y)$ and $\cos(y)$ are just the functions as defined for real numbers.

18) What is the radius and argument of $e^{x+iy}$ when $x, y \in \mathbb{R}$?

The radius is $e^x$ and the argument is $y + 2\pi k$ where $k$ is any integer.

19) What is De Moivre’s formula?

De Moivre’s Formula says that since $(e^{i\theta})^n = e^{i\theta}$ then $(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$.

Section 1.5: Powers and Roots

20) What are the $n$th roots of unity for $n = 1, 2, 3, 4, \ldots$?

They are the elements $z$ of $\mathbb{C}$ such that $z^n = 1$. Specifically they are $e^{2\pi ik/n}$ where $k = 0, 1, 2, \ldots, n-1$ and these points are the corners of an $n$-side regular polygon.

21) If $z = re^{i\theta}$, what is $z^{1/m}$ for $m = 1, 2, 3, 4, \ldots$?

It is just the set of $m$th roots of $z$, $\sqrt[m]{re^{i\theta}} e^{2\pi ik/m}$ where $m = 0, 1, 2, \ldots, m - 1$.

22) What are the zeros of $1 + z + z^2 + \cdots z^m$ for $m = 1, 2, 3, 4, \ldots$?

The $m$th roots of unity, $\omega_m$, $\omega_m^2$, $\ldots$, $\omega_m^{m-1}$, except for $z = 1$.

Section 1.6: Planar sets

23) Give an algebraic description for the open disk of radius $\sqrt{3}$ centered at $1 + 2i$.

Just the set of $z \in \mathbb{C}$ where $|z - 1 - 2i| < \sqrt{3}$.

24) Give an algebraic description for the closed disk of radius $\sqrt{3}$ centered at $1 + 2i$.

Just the set of $z \in \mathbb{C}$ where $|z - 1 - 2i| \leq \sqrt{3}$.

25) What is an open set?

A set where every point in the set is an interior point, where for every point, $z_0$, in the set an open disk centered at $z_0$ is also contained in the set if the disk is sufficiently small.

26) What is a connected set?
A set where any two points can be connected by a path that is also contained in the set.

27) What is a domain?
An open, connected set.

28) What is a bounded set?
A set that can be entirely contained in a sufficiently large open disk. Alternately, a set where if you “zoomed out” far enough you would see the entire set.

29) If a function \( f(z) \) has zero first partial derivatives \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) on all of a domain, \( D \), then what can we say about \( f(z) \)?
We can say \( f(z) \) is constant on \( D \).

30) If two functions \( f(z) \) and \( g(z) \) have the same first partial derivatives \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) on all of a domain, \( D \), then what can we say about the relationship between \( f(z) \) and \( g(z) \)?
We can say that \( f(z) = g(z) + C \) on \( D \) for some constant \( C \in \mathbb{C} \).

Chapter 2: Analytic Functions

Section 2.1: Functions of a Complex Variable
31) What is the range and domain of definition of a function?
The domain of definition is the set of points where a function is defined (inputs), and is not the same concept as a domain (open, connected set). A range is the set of values (outputs) of a function.

32) What is the domain of definition for \( f(z) = \log(z) \)?
All of \( \mathbb{C} \) except for \( z = 0 \). [But note that \( f(z) \) is only analytic for all \( \mathbb{C} \) except for nonpositive real values.]

Section 2.2: Limits and Continuity
33) What does it mean for a sequence \( \{z_n\} \) to have a limit, \( z_0 \)?
We can take \( z_n \) to be arbitrarily close (in terms of distance) to \( z_0 \) if \( n \) is large enough. Or, put simply (if less accurately) if \( n \) gets big then \( z_n \) gets close to \( z_0 \).

34) What does \( \lim_{z \to z_0} f(z) = w \) mean?
Put simply (if less accurately), as $z$ gets close to $z_0$, $f(z)$ gets close to $w$. “Closeness” is defined in terms of distance.

35) What does it mean for $f(z)$ to be continuous at $z_0$ or on a domain $D$?

For $f(z)$ to be continuous at a point, $z_0$, it means that $\lim_{z \to z_0} f(z) = f(z_0)$, that is the limit is equal to the value of the function. For $f(z)$ to be continuous on $D$ it means it is continuous at every point in $D$.

36) If $f(z)$ and $g(z)$ are continuous on a domain $D$, when is $f(z)/g(z)$ not continuous?

Only possibly when $g(z) = 0$.

37) What does it mean that $f(z)$ has a removable discontinuity at $z_0$?

It means $\lim_{z \to z_0} f(z)$ exists but is not $f(z_0)$ or $f(z_0)$ is not defined. We can “remove” this discontinuity by letting $f(z)$ be the value of the limit.

Section 2.3: Analyticity

38) How do we write $f(x, y)$ as a function $f(z)$?

Let $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$.

39) What is an “admissible function” and why do we care about them?

An admissible function is a function that can be written only in terms of $z$ without $\overline{z}$. We care about these because they are generally analytic.

40) What does it mean for $f(z)$ to be differentiable at $z_0$?

It means the limit $\lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$ exists, and $f'(z_0)$ is that limit.

41) What can we say about the differentiability of $f(z) = \overline{z}$.

It is nowhere differentiable.

42) What does it mean for a function $f(z)$ to be analytic?

It means that $f(z)$ is differentiable for every point on an open connected set.

43) What does it mean for a function $f(z)$ to be analytic at a point $z_0$?

It means $f(z)$ is analytic on a domain containing $z_0$.

Section 2.4: The Cauchy-Riemann Equations
44) What are the Cauchy-Riemann Equations?

Put simply, \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \).

45) What do the CREs have to do with differentiability of \( f(z) = u(x, y) + iv(x, y) \)?

If all first partials of \( u \) and \( v \) exist on an open set containing \( z_0 \), are continuous at \( z_0 \), and the CREs are satisfied at \( z_0 \) then \( f(z) \) is differentiable at \( z_0 \).

46) What can we say about \( f(z) \) if \( f'(z) = 0 \) on a domain \( D \)?

We can say \( f(z) \) is constant on \( D \).

47) What can we say about \( f(z) \) and \( g(z) \) if \( f'(z) = g'(z) \) on a domain \( D \)?

That \( f(z) = g(z) + C \) on \( D \) for some constant \( C \in \mathbb{C} \).

48) Write \( f'(z) \) in terms of partial derivatives with respect to \( x \) and with respect to \( y \).

We have \( f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x} \). We see these representations are the same by the Cauchy-Riemann Equations (and is how we derived the CREs in the first place).

Section 2.5: Harmonic Functions

49) What is the Laplace Equation?

Just that \( \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) for some function of \( \mathbb{R}^2 \).

50) What does it mean that \( u(x, y) \) is harmonic on a domain \( D \)?

It means that \( u \) satisfies the Laplace Equation on \( D \).

51) What is the relationship between harmonic and analytic functions?

If \( f(z) \) is analytic then \( \text{Re}(f(z)) \) and \( \text{Im}(f(z)) \) are harmonic.

52) If \( \phi(x, y) \) is harmonic on a domain, \( D \), what does it mean that \( \psi(x, y) \) is a harmonic conjugate of \( \phi(x, y) \)?

It means \( f(z) = f(x + iy) = \phi(x, y) + i\psi(x, y) \) is analytic on \( D \).

Chapter 3: Elementary Functions

Section 3.1: Polynomials and Rational Functions

53) How many zeros/roots does a polynomial of degree \( n \) have?

By the fundamental theorem of algebra, \( n \).
54) What is the Taylor form of a polynomial $f(z)$ of degree $m$ at $z = z_0$?

It’s just

$$f(z) = \sum_{n=0}^{m} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2} (z - z_0)^2 + \cdots + \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m.$$  

55) If $f(z) = \frac{A}{(z-1)^7} + \frac{B}{(z-2)^6} + \frac{C}{(z-2)^2}$, what is a formula for $C$ in terms of limits and derivatives of $f(z)$?

We have $C = \lim_{z \to 2} \frac{1}{4!} \frac{d^4}{dz^4} [(z - 2)^6 f(z)].$

Section 3.2: The Exponential, Trigonometric, and Hyperbolic Functions

56) Is $f(z) = e^z$ one-to-one, and why or why not?

No, because $f(z_1) = f(z_2)$ exactly when $z_2 = z_1 + 2\pi ik$ for $k \in \mathbb{Z}$.

57) What is a fundamental region of a function $f(z)$?

A region where $f(z)$ is one-to-one and $f(z)$ achieves every possible value?

58) Is the set $0 \leq \text{Im}(z) \leq 2\pi$ a fundamental region for $e^z$?

No, because $0$ and $2\pi i$ are both in this region and $e^0 = e^{2\pi i}$. However, $0 \leq \text{Im}(z) < 2\pi$ is a fundamental region for $e^z$.

59) Give the definition of $\sin(z)$, $\cos(z)$, $\sinh(z)$, and $\cosh(z)$ for complex $z$.

They are all fairly similar:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sinh(z) = \frac{e^z - e^{-z}}{2}, \quad \cosh(z) = \frac{e^z + e^{-z}}{2}.$$  

Section 3.3: The Logarithmic Function

60) What is log($z$) for $z \neq 0$?

It is the infinite set $\ln(|z|) + i \text{arg}(z) = \text{Log}(|z|) + i \text{Arg}(z) + 2\pi ik$, the set of values, $w$, such that $e^w = z$.

61) Why is log($z$) infinitely valued?

The imaginary part of log($z$) is the argument of $z$, which has infinitely many values.
62) How can we simplify \( \log(z_1) + \log(z_2) \)?

It’s just \( \log(z_1z_2) \).

63) What is the principal branch of logarithm, \( \log(z) \)?

It is \( \log(z) = \ln(|z|) + i\text{Arg}(z) \), which we specify by specifying a branch of argument, \( \text{Arg}(z) \). Since \( \log(|z|) \) agrees with \( \ln(|z|) \) we often use \( \log \) instead.

64) What is \( L_\tau(z) \)?

An alternate branch of log given by an alternate branch of argument, \( L_\tau(z) = \log(|z|) + i\text{arg}_\tau(z) \).

65) What is \( \frac{d}{dz}L_\tau(z) \)?

It is \( \frac{1}{z} \) except where \( L_\tau(z) \) has a branch cut and is thus not continuous or differentiable.

66) Is it always true that \( \log(z_1z_2) = \log(z_1) + \log(z_2) \)?

No, the branch cut gets in the way. For example \( \log(-1) = i\pi \) and \( \log(-i) + \log(-i) = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi \), but \((-i)(-i) = 1 \).

67) Why is \( u(x,y) = \log(|z|) \) harmonic everywhere except \( z = 0 \)?

We know \( u(x,y) \) is the real part of \( \log(z) \) and \( L_0(z) \), and so must be harmonic anywhere either of those two functions are analytic, which is everywhere but \( z = 0 \). It is not harmonic at \( z = 0 \) because \( \log(z) \) is undefined at \( z = 0 \).

68) Where is \( \text{arg}_{\frac{\pi}{2}}(z) \) harmonic and why?

It is harmonic everywhere except for the nonnegative part of the imaginary axis. This is because \( L_{\frac{\pi}{2}}(z) \) is analytic everywhere except for the branch cut on the nonnegative part of the imaginary axis and \( \text{arg}_{\frac{\pi}{2}}(z) \) is the imaginary part of \( L_{\frac{\pi}{2}}(z) \).

Section 3.4: Washers, Wedges, and Walls

69) What harmonic functions do we use to solve constant-boundary value problems on washers?

We use linear combinations of \( \log|z - \alpha| \) for some \( \alpha \in \mathbb{C} \) and the constant function.

70) What harmonic functions do we use to solve constant-boundary value problems on wedges and walls?

We use linear combinations of branches of \( \text{arg}(z - \alpha) \) and the constant function.
71) What harmonic functions do we use to solve the constant-boundary value problem on disks?

We use constant functions.

Section 3.5: Complex Powers

72) What does $z^\alpha$, for $z \neq 0$, mean?

It is the set of points given by $e^{\alpha \log(z)}$.

73) When is $z^\alpha$, for $z = 0$, single-valued, finitely-valued and infinitely valued?

We know that $z^\alpha$ is single valued if $\alpha$ is an integer, finitely valued when $\alpha$ is rational, and infinitely valued when $\alpha$ is irrational. More specifically, if $\alpha = m/n$, a rational number in reduced form, then $z^\alpha$ has $n$ distinct values.

74) What is the principal branch of $z^\alpha$ for $z \neq 0$?

It is $e^{\alpha \text{Log}(z)}$, that is we use the principal branch of log in our definition of $z^\alpha$.

75) What does it mean to have a branch, $F(z)$, of a multi-valued function, $f(z)$ on a domain $D$?

It means that on $D$, the function $F(z)$ is continuous and agrees with one of the values of $f(z)$ on $D$.

Chapter 4: Complex Integration

Section 4.1: Contours

76) Geometrically, what does it mean for a curve/arc to be smooth?

It means that it has no corners or cusps (places where the derivative $\gamma'(t)$ is zero or undefined) and does not self-intersect except perhaps at the endpoints (one-to-one).

77) What does it mean for a curve to be closed?

The endpoints are the same.

78) What is a parametrization of the part of the circle, centered at $3i$ with radius 7, with traversed clockwise from argument 0 and $\pi/4$ relative to the center of the circle?

We can use $\gamma(t) = 3i + 7e^{i\theta}$ where $\theta \in [0, \pi/4]$, though there are other parametrizations.

79) What is a parametrization of the directed line segment from $2 + i$ to $\pi$?
We can use \( \gamma(t) = (2 + i)(1 - t) + \pi(t) \) for \( t \in [0, 1] \) though there are other parametrizations.

80) If \( \gamma_1, \gamma_2, \ldots, \gamma_m \) are directed curves where the endpoint of \( \gamma_k \) is the starting point of \( \gamma_{k+1} \) for \( k = 1, 2, \ldots, m - 1 \), what does it mean that \( \Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_m \)?

It means that \( \Gamma \) is the contour that traces \( \gamma_1(t) \) then \( \gamma_2(t) \) all through \( \gamma_m(t) \) in order.

81) If \( \Gamma \) is a directed contour, what is \(-\Gamma\)?

It is the same contour traversed in the opposite direction.

82) What is Jordan’s Curve Theorem?

Any simple (not otherwise self-intersecting) closed contour \( \Gamma \) divides \( \mathbb{C} \) into two domains that have \( \Gamma \) as a boundary, a bounded region called the interior and an unbounded region called the exterior.

83) What does it mean for a simple contour to be positively or negatively oriented?

When following the direction of the contour, if the interior of the contour is to the left then the contour is positively oriented. Otherwise it is negatively oriented.

84) What is the formula for the length of the smooth curve \( \gamma'(t) \) for \( t \in [a, b] \).

It is just \( \int_a^b |\gamma'(t)| \, dt \).

Section 4.2: Contour Integrals

85) Given a continuous, complex-valued function, \( f(t) \), for a real variable \( t \), what does \( \int_a^b f(t) \, dt \) mean?

It is just means \( \int_a^b \text{Re}(f(t)) \, dt + i \int_a^b \text{Im}(f(t)) \, dt \), where each of the integrals are just real integrals like the ones we’ve seen in calculus.

86) Given a smooth curve \( \gamma(t) \) for \( t \in [a, b] \) and \( f(z) \) continuous on \( \gamma \), what is the formula for \( \int_\gamma f(z) \, dz \) and, generally, what does it mean?

It is \( \int_a^b f(\gamma(t)) \gamma'(t) \, dt \), which we know how to integrate as the integral of a complex-valued function on a single variable. Generally, this integral “adds up” the values of \( f(z) \) along \( \gamma \).

87) If \( C_r \) is a positively oriented circle of radius \( r \) centered at \( z_0 \), what is \( \oint_{C_r} (z - z_0)^n \, dz \) for \( n \in \mathbb{Z} \)?

It is zero if \( n \neq -1 \) and \( 2\pi i \) if \( n = -1 \).
88) If $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_m$, where each $\gamma_k$ is a smooth curve, what does $\int_{\Gamma} f(z) \, dz$ mean in terms of these $\gamma_k$ and what does $\int_{-\Gamma} f(z) \, dz$ mean in terms of $\int_{\Gamma} f(z) \, dz$?

We say $\int_{\Gamma} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz + \cdots + \int_{\gamma_m} f(z) \, dz$ and $\int_{-\Gamma} f(z) \, dz = -\int_{\Gamma} f(z) \, dz$.

89) If $|f(z)| \leq M$ on the contour $\gamma$, give an upper bound for $|\int_{\gamma} f(z) \, dz|$.

We have $|\int_{\gamma} f(z) \, dz| \leq \ell(\gamma)M$ where $\ell(\gamma)$ is the length of $\gamma$.

Section 4.3: Independence of Path

90) If $f(z)$ has an anti-derivative, $F(z)$, on a domain containing the contour $\Gamma$ with endpoints $a$ and $b$ then what is $\int_{\Gamma} f(z) \, dz$?

It is just $F(b) - F(a)$.

91) If $f(z)$ has an anti-derivative on a domain containing $\gamma$ a closed contour, $\Gamma$, then what is $\int_{\Gamma} f(z) \, dz$?

It is zero, since the endpoints are the same.

92) Why are contour integrals of functions with anti-derivative called “path-independent”?

The integral can be evaluated via the anti-derivative at the endpoints and so the path between the endpoints does not matter.

93) Let $f(z)$ be continuous on the domain $D$. If for any two contours, $\Gamma_1$ and $\Gamma_2$ in $D$ with the same endpoints we have $\int_{\Gamma_1} f(z) \, dz = \int_{\Gamma_2} f(z) \, dz$ then what can we say about $f(z)$?

We can say that $f(z)$ has an anti-derivative on $D$.

94) How do we know that $f(z) = 1/z$ does not have an antiderivative on all of $\mathbb{C}$ except the origin?

Because we know $\oint_{C_r} z^{-1} \, dz = 2\pi i$ for any positively oriented circle, $C_r$, centered at 0, and if an anti-derivative existed on all of $\mathbb{C} - \{0\}$ then this integral would be zero.

Section 4.4: Cauchy’s Integral Theorem

95) In plain language, what does it mean to continuously deform a closed contour in a domain $D$?

It means you can bend and move the contour as long as all intermediate steps stay inside the domain $D$.

96) What does it mean for a domain, $D$, to be simply connected?
It means that any closed contour in \( D \) can be continuously deformed to a point in the domain. Alternately, it means that the domain \( D \) does not surround any holes.

97) What is the Deformation Invariance Theorem?

If \( f(z) \) is analytic on a domain \( D \) and the loops \( \Gamma_1 \) and \( \Gamma_2 \) can be continuously deformed into each other in \( D \) then \( \int_{\Gamma_1} f(z) \, dz = \int_{\Gamma_2} f(z) \, dz \).

98) What is Cauchy’s Integral Theorem and how does it follow from the Deformation Invariance Theorem?

Cauchy’s Integral Theorem says that for a simply connected domain \( D \) where \( f(z) \) is analytic then \( \int_{\Gamma} f(z) \, dz = 0 \) for any closed loop \( \Gamma \) contained in \( D \). This follows from the Deformation Invariance Theorem because if \( D \) is simply connected then \( \Gamma \) can be continuously deformed to a point \( \{z_0\} \) in \( D \) and so \( \int_{\Gamma} f(z) \, dz = \int_{\{z_0\}} f(z) \, dz = 0 \).

99) What does Cauchy’s Integral Theorem say about analytic functions, path independence, and anti-derivatives?

Cauchy’s Integral Theorem says that on simply connected domains contour integrals of analytic functions are path independent which means that analytic functions on simply connected domains have anti-derivatives.

Section 4.5: Cauchy’s Integral Formula and Its Consequences

100) What is Cauchy’s Integral Formula?

Cauchy’s Integral formula says that if \( f(z) \) is analytic on a domain containing the simple, positively oriented, closed contour \( \Gamma \) and its interior and \( z_0 \) is in the interior of \( \Gamma \) then \( f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(w)}{w-z_0} \, dw \).

101) What is the generalized Cauchy’s Integral Formula?

The generalized Cauchy’s Integral formula says that if \( f(z) \) is analytic on a domain containing the simple, positively-oriented closed contour \( \Gamma \) and its interior and \( z_0 \) is in the interior of \( \Gamma \) then \( f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(w)}{(w-z_0)^{n+1}} \, dw \).

102) What does the generalized Cauchy’s Integral Formula say about the differentiability of \( f'(z) \) if \( f(z) \) is analytic?

It says that if \( f(z) \) is analytic on a domain \( D \) it is infinitely differentiable on \( D \), that is all derivatives \( f'(z), f''(z), f'''(z), \ldots \) exist.

103) What does the generalized Cauchy’s Integral Formula say about the partial derivatives of a harmonic function \( u(x,y) \)?
It says that if \( u(x, y) \) is harmonic on a domain \( D \) then all higher-order partial derivatives of \( u(x, y) \) exist.

104) What is Morera’s Theorem?

Let \( f(z) \) be continuous on the domain \( D \) and \( \oint_{\Gamma} f(z) \, dz = 0 \) for any closed loop \( \Gamma \) in \( D \) then \( f(z) \) is analytic in \( D \).

Section 4.6: Bounds for Analytic Functions

105) Suppose \( f(z) \) is analytic on a circle centered \( C_R \), centered at \( z_0 \) with radius \( R \), and its interior, and suppose \( |f(z)| \leq M \) for \( z \in C_R \). What can we say about \( f^{(n)}(z_0) \)?

We can say \( |f^{(n)}(z_0)| \leq \frac{n!M}{R^n} \).

106) What is Liouville’s Theorem?

Liouville’s Theorem says that all bounded entire functions are constant.

107) What is the maximum modulus principle?

It says that given a domain \( D \) and \( f(z) \) which is analytic on that domain, if there exists a point \( z_0 \) such that \( |f(z_0)| \geq |f(z)| \) for all \( z \in D \) then \( f(z) \) is constant on \( D \). It also says that if \( f(z) \) is analytic on a bounded domain \( D \) and is continuous on the domain and the boundary, then \( |f(z)| \) attains its maximum for \( z \) on the boundary.

Section 4.7: Applications to Harmonic Functions

108) What can we say about harmonic functions on a simply connected domain \( D \)?

They are the real part of an analytic function on \( D \).

109) What is the Maximum-Minimum principle for harmonic functions?

It says that given a domain \( D \) and \( u(x, y) \) which is harmonic on that domain, if there exists a point \( (x_0, y_0) \) such that \( u(x_0, y_0) \geq u(x, y) \) for all \( (x, y) \in D \) or \( u(x_0, y_0) \leq u(x, y) \) for all \( (x, y) \in D \) then \( u(x, y) \) is constant on \( D \). It also says that if \( u(x, y) \) is harmonic on a bounded domain \( D \) and is continuous on the domain and the boundary, then \( u(x, y) \) attains its maximum and minimum on the boundary.

110) What is the Dirichlet Problem?

The Dirichlet Problem is: Find a function \( \phi(x, y) \) that is continuous on a domain \( D \) and its boundary, harmonic in \( D \), and taking specified values on the boundary of \( D \).
111) Given two harmonic functions, \( \phi_1 \) and \( \phi_2 \) on the bounded domain \( D \) that are continuous up to the boundary of \( D \) and \( \phi_1 = \phi_2 \) on the boundary, what can we say about \( \phi_1 \) and \( \phi_2 \) and why?

We can say \( \phi_1 = \phi_2 \) because \( \phi_1 - \phi_2 \) is harmonic on \( D \) and zero on the boundary so the maximum-minimum principle says that \( \phi_1 - \phi_2 = 0 \) on \( D \).

112) What is Poisson’s Integral formula?

If \( U(x, y) \) is continuous on the circle \( |z| = R \) except for finitely many discontinuities then there exists \( u(x, y) \) which is harmonic on \( |z| < R \) and continuous on \( |z| \leq R \), except where \( U(x, y) \) is discontinuous, and agrees with \( U(x, y) \) on \( |z| = R \) except where \( U(x, y) \) is discontinuous. For any point \( z = re^{i\theta} \) in the disk \( |z| < R \) we have

\[
u(re^{i\theta}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{U(Re^{it})}{R^2 + r^2 - 2rR \cos(t-\theta)} \, dt.
\]

Chapter 5: Series Representations for Analytic Functions

Section 5.1: Sequences and Series

113) What does it mean to say a series \( \sum_{k=0}^{\infty} a_k \), where \( a_k \in \mathbb{C} \), converges to \( L \)?

It means the sequence of partial sums, \( \sum_{k=0}^{n} a_k \), converges to \( L \) as \( n \to \infty \).

114) What is the geometric series, when does it converge, and what does it converge to?

The geometric series is \( \sum_{n=0}^{\infty} c^n \) which converges to \( \frac{1}{1-c} \) exactly when \( |c| < 1 \).

115) What is the comparison test?

If \( |a_k| \leq M_k \) for \( M_k \in \mathbb{R}_{>0} \) and \( \sum_{k=0}^{\infty} M_k \) converges then \( \sum_{k=0}^{\infty} a_k \) converges.

116) When does the \( p \)-series converge?

The series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) where \( p \) is real converges exactly when \( p > 1 \).

117) What is the ratio test?

The ratio test says that if \( |c_{k+1}/c_k| \to L \) as \( k \to \infty \) then the series \( \sum_{k=0}^{\infty} c_k \) converges if \( L < 1 \) or diverges if \( L > 1 \). If \( L = 1 \) then the test tells us nothing.

Section 5.2: Taylor Series

118) For a function \( f(z) \) that is analytic at \( z_0 \), what is its Taylor series and where does it converge?
The Taylor series is $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$ and it converges on the largest open disk centered at $z_0$ where $f(z)$ is analytic.

119) What is the Maclaurin Series of $f(z)$?
It is the Taylor Series of $f(z)$ at $z = 0$, should this series exist.

120) If $f(z)$ has the Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ at $z_0$, what is the Taylor series of $f'(z)$?
It is $f(z) = \sum_{n=1}^{\infty} n a_n(z-z_0)^{n-1}$.

121) If $f(z)$ and $g(z)$ are analytic at $z_0$ with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ then what is the Taylor series of $f(z) + g(z)$ at $z_0$?
It is $f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z-z_0)^n$.

Section 5.3: Power Series
122) What is a power series?
A power series is any series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ that is convergent on some open disk centered at $z_0$. The series is a function of $z$ that is analytic on any such disk.

123) What is the radius of convergence of a power series centered at $z_0$?
The radius of convergence is the largest $R$ such that the power series converges on the open disk $|z-z_0| < R$.

124) What is the relationship between Taylor Series and Power Series.
The Taylor Series is an example of a power series and a power series is actually its own Taylor Series, since it is analytic. So really they are the same, but usually a Taylor series begins with an analytic function.

125) If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is convergent on some open disk centered at $z_0$, what is $f^{(m)}(z_0)$?
It is $a_m m!$.

Section 5.5: Laurent Series
126) What is a Laurent series expansion of $f(z)$ and where is it defined?
A Laurent series expansion of $f(z)$ is a series expansion of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ which is defined on an annulus, $R > |z-z_0| > r$ where $\infty \geq R > r \geq 0$, where $f(z)$ is analytic.
127) If \( \sum_{n=1}^{\infty} a_n (z - z_0)^{-n} \) converges for \( |z - z_0| > r \) and \( \sum_{n=0}^{\infty} a_n (z - z_0) \) converges for \( |z - z_0| < R \) then where does \( \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \) converge?

On the annulus \( R > |z - z_0| > r \).

Section 5.6: Zeros and Singularities

128) What does it mean for an analytic function to have a zero of order \( m \) at \( z_0 \)?

It means \( f(z) \) has the Taylor series \( \sum_{n=m}^{\infty} a_n (z - z_0)^n \) where \( a_m \neq 0 \). Alternately it means \( f(z) = (z - z_0)^m g(z) \) where \( g(z) \) is analytic at \( z_0 \) and \( g(z_0) \neq 0 \).

129) What does it mean for a function to have a pole of order \( m \) at \( z_0 \)?

It means \( f(z) \) has the Laurent series \( \sum_{n=-m}^{\infty} a_n (z - z_0)^n \) where \( a_{-m} \neq 0 \) on the punctured disk \( r > |z - z_0| > 0 \). Alternately it means \( f(z) = (z - z_0)^{-m} g(z) \) where \( g(z) \) is analytic at \( z_0 \) and \( g(z_0) \neq 0 \).

130) What is it mean if \( z_0 \) is an isolated singularity of \( f(z) \)?

It means that \( f(z) \) is analytic on a punctured disk centered at \( z_0 \) but is not at \( z_0 \) itself.

131) What is it mean if \( z_0 \) is a removable singularity of \( f(z) \)?

It means that \( z_0 \) is an isolated singularity of \( f(z) \) but that if we redefine \( f(z_0) \) to be \( \lim_{z \to z_0} f(z) \) then it will be analytic. It also means that the Laurent Series of \( f(z) \) on a punctured disk centered at \( z_0 \) has no negative powers of \( (z - z_0) \) in it.

132) What is it mean if \( z_0 \) is an essential singularity of \( f(z) \)?

It means that the Laurent Series of \( f(z) \) on a punctured disk centered at \( z_0 \) has infinitely many negative powers of \( (z - z_0) \) in it.

133) If \( f(z) \) has a zero of order \( m \) at \( z_0 \), what can we say about \( 1/f(z) \) at \( z_0 \)?

It has a pole of order \( m \) at \( z_0 \).

Chapter 6: Residue Theory

Section 6.1: The Residue Theorem

134) If \( z_0 \) is an isolated singularity of \( f(z) \), what is \( \text{Res}(f, z_0) \), the residue of \( f \) at \( z_0 \)?

Given the Laurent Series of \( f(z) \), \( \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \) on the punctured disk centered at \( z_0 \), the residue of \( f \) at \( z_0 \) is just \( a_{-1} \).

135) If \( z_0 \) is a pole of order \( m \) of the function \( f(z) \), what is the formula for \( \text{Res}(f, z_0) \)?
It is \( \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \).

136) What is Cauchy’s Residue Theorem?

Cauchy’s Residue Theorem states that if \( f(z) \) is analytic on a simple, positively oriented closed contour \( \Gamma \) and its interior except for isolated singularities at \( z_1, z_2, \ldots, z_m \) then \( \oint_{\Gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{m} \text{Res}(f, z_k) \).

Section 6.2: Trigonometric Integrals over \([0, 2\pi]\)

137) If \( U(x, y) \) is a rational function in \( x \) and \( y \), how can we represent \( \int_{0}^{2\pi} U(\cos(\theta), \sin(\theta)) \, d\theta \) as an integral on a closed contour of a rational function in \( z \)?

It is just \( \oint_{|z|=1} U(\frac{e^{i\theta}+e^{-i\theta}}{2}, \frac{e^{i\theta}-e^{-i\theta}}{2i}) \, \frac{dz}{iz} \) which we get by taking the substitution \( e^{i\theta} = z \).

138) What is the advantage of writing \( \int_{0}^{2\pi} U(\cos(\theta), \sin(\theta)) \, d\theta \) as an integral on a closed contour of a rational function in \( z \)?

It is very straightforward to use Cauchy’s Residue Thereom (or Cauchy’s Generalized Integral Formula) to compute integrals on closed contours of rational functions in \( z \).

Section 6.3: (Optional) Improper Integrals

139) Suppose \( P(x) \) and \( Q(x) \) are polynomials with the degree of \( P(x) \) at least two less than the degree of \( Q(x) \). Suppose further that \( Q(z) \) has zeros \( z_1, z_2, \ldots, z_m \) in the upper half plane (above the real axis) and zeros \( w_1, w_2, \ldots, w_n \) in the lower half plane (below the real axis) and no zeros on the real axis. What is \( \text{p.v.} \int_{-\infty}^{\infty} f(x) \, dx \) where \( f(z) = \frac{P(z)}{Q(z)} \)?

It is \( 2\pi i \sum_{k=1}^{m} \text{Res}(f, z_k) \) or \( -2\pi i \sum_{k=1}^{n} \text{Res}(f, w_k) \).

140) Roughly, what is the outline of the argument for computing the integral in the previous problem?

Since \( \text{p.v.} \int_{-\infty}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{-t}^{t} f(x) \, dx \), we can consider the integral of \( f(z) \) on a contour that traces the real line segment \([-t, t] \) and then the counterclockwise semicircle \( te^{i\theta} \) for \( \theta \in [0, \pi] \). As \( t \) gets large, Cauchy’s Residue Theorem gives us that the integral on the contour eventually becomes \( 2\pi i \) times the sum of residues in the upper half plane and the integral along the semicircle goes to zero, giving the result. We can use the same argument for the lower half plane, but the curve will be clockwise so we get \(-2\pi i\) times the sum of residues in the lower half plane instead.