I. Cayley Graphs

$x = a^{-1}b$

$b = ax$

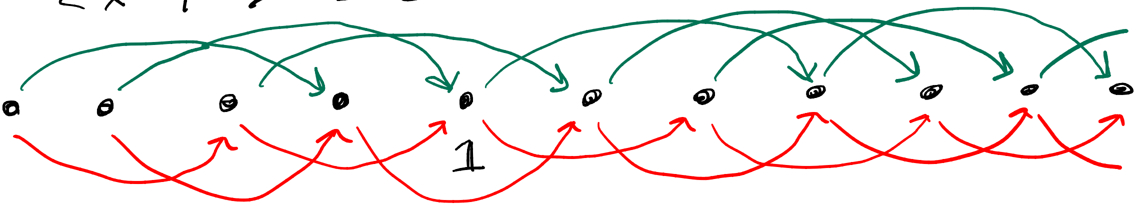
$b^2 = a^2x^2$

$a^3 = a^2x^2$

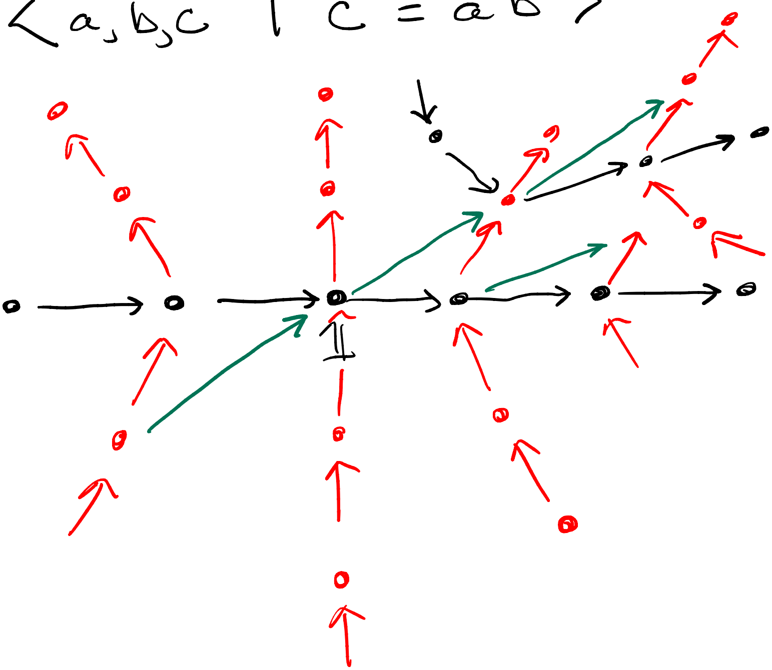
$a = x^2$

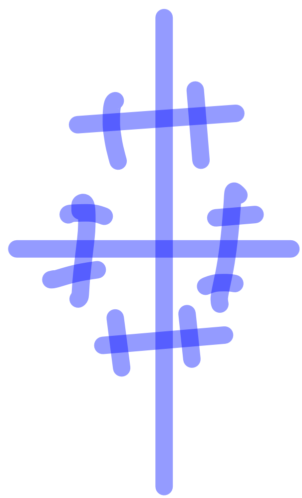
$\langle a, b \mid a^3 = b^2, [a,b] \rangle$

$= \langle x \mid \rangle \cong \mathbb{Z}$



$\langle a, b, c \mid c = ab \rangle \cong \mathbb{F}_2$





II Where does the definition of "Quasi-Isometry" come from?

Consider a group G w/ finite gen sets S, S'

Ex $S = \{a, b, a^{-1}, b^{-1}\}$

$S' = \{x, y, z, x^{-1}, y^{-1}, z^{-1}\}$

$x =$ WORD IN a, b, a^{-1}, b^{-1}

$= s_1 s_2 s_3 \cdots s_{n_x}$

$y = t_1 t_2 \cdots t_{n_y} =$ WORD IN a, b, a^{-1}, b^{-1}

$z = u_1 u_2 \cdots u_{n_z} =$ WORD IN a, b, a^{-1}, b^{-1}

A word of length n in $x^{\pm}, y^{\pm}, z^{\pm}$ can be written as a word of length at most $n \cdot \max(n_x, n_y, n_z)$ in $a^{\pm}, b^{\pm}$

$x\,y\,z = s_1 s_2 s_3 \cdots s_{n_x} t_1 t_2 \cdots t_{n_y} u_1 u_2 \cdots$

<u>Lemma</u> $\exists$ constant $K_1 \geq 1$ s.t.

$$d_{S'}(g, \mathbb{1}) \leq K_1\, d_S(g, \mathbb{1})$$

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$$d_{S'}(g, \mathbb{1}) \leq K_1 \, d_S(g, \mathbb{1})$$

$\exists$ constant $K_2 \geq 1$ s.t.

$$d_S(g, \mathbb{1}) \leq K_2 \, d_{S'}(g, \mathbb{1})$$

$$\Rightarrow \quad \frac{1}{K_2} d_S(g, \mathbb{1}) \leq d_{S'}(g, \mathbb{1}) \leq K_1 \, d_S(g, \mathbb{1})$$

Let $K = \max(K_1, K_2)$

$$\Rightarrow \quad \frac{1}{K} d_S(g, \mathbb{1}) \leq d_{S'}(g, \mathbb{1}) \leq K \, d_S(g, \mathbb{1})$$

Notice $d_S(g, h) = d_S(\mathbb{1}, g^{-1}h)$

and $d_{S'}(\quad) = d_{S'}(\quad)$

$\Rightarrow$ <u>prelim Thm</u> $\exists$ constant K (depends on S, S') s.t.

$$\frac{1}{K} d_S(g, h) \leq d_{S'}(g, h) \leq K \, d_S(g, h)$$

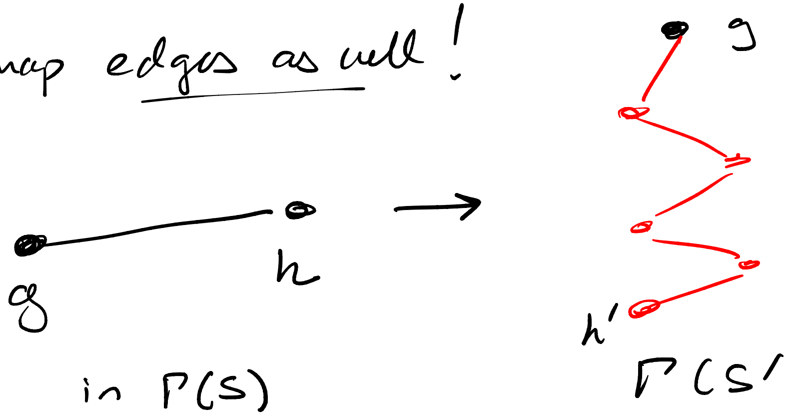
Cayley graph

So we can map vertices of $\Gamma(S) \rightarrow$ vertices for $\Gamma(S')$

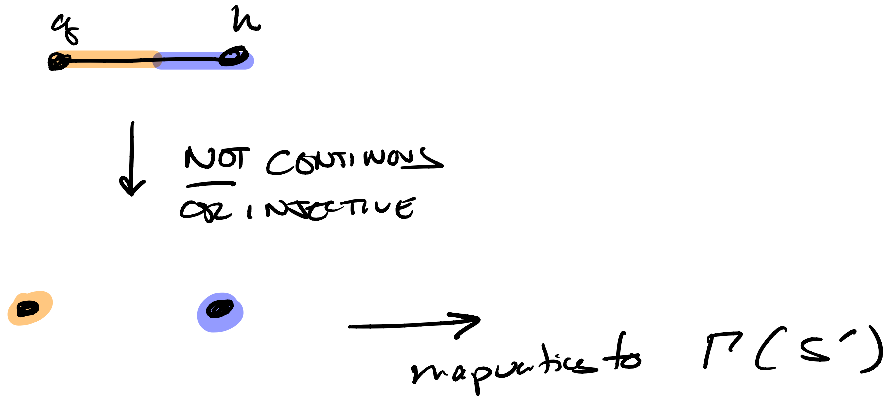
w/ <u>bounded</u> stretching / shrinking

Want to map edges as well!

BUT

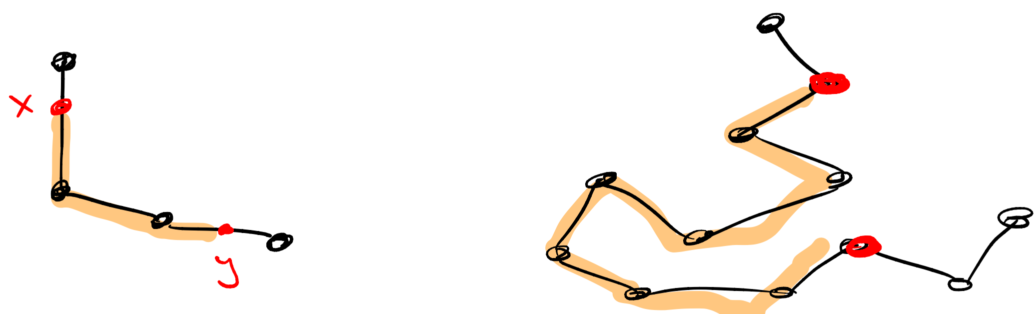


So



Send all of $\Gamma(S) \to$ vertices of $\Gamma(S')$

So if x, y are interior to edges of $\Gamma(S)$



<u>Thm</u> If G has finite gen. sets S, S' then there exists a function $f: \Gamma(S) \to \Gamma(S')$ s.t. $\exists$ constants $K \geq 1$, $C \geq 0$ s.t. $\forall\ x, y \in \Gamma(S)$

$$\frac{1}{K} d_S(x,y) - C \leq d_{S'}(f(x), f(y)) \leq K d_S(x,y) + C$$

"Bounded scaling and Bounded tearing"

III • A function $f:(X,d_X)\to(Y,d_Y)$ is a quasi-isometric embedding if $\exists$ constants $K\geq 1$ $C\geq 0$ s.t.

$$\frac{1}{K}d_X(x,y)-C \leq d_Y(f(x),f(y)) \leq K\,d_X(x,y)+C$$

Ex $f:\mathbb{Z}\to\mathbb{R}$ $f(n)=n$

$g:\mathbb{R}\to\mathbb{Z}$ $g(x)=\lfloor x\rfloor$ $K=1$ $c=1$

• A function $f:(X,d_X)\to(Y,d_Y)$ is a quasi-isometry if $\exists D>0$ s.t. $\forall y\in Y$ $d_Y(y,f(X))\leq D$ & f is a quasi-isom. embedding.

"coarsely surjective" : "quasi-dense"
: "every pt in Y is a bounded distance from image of X"

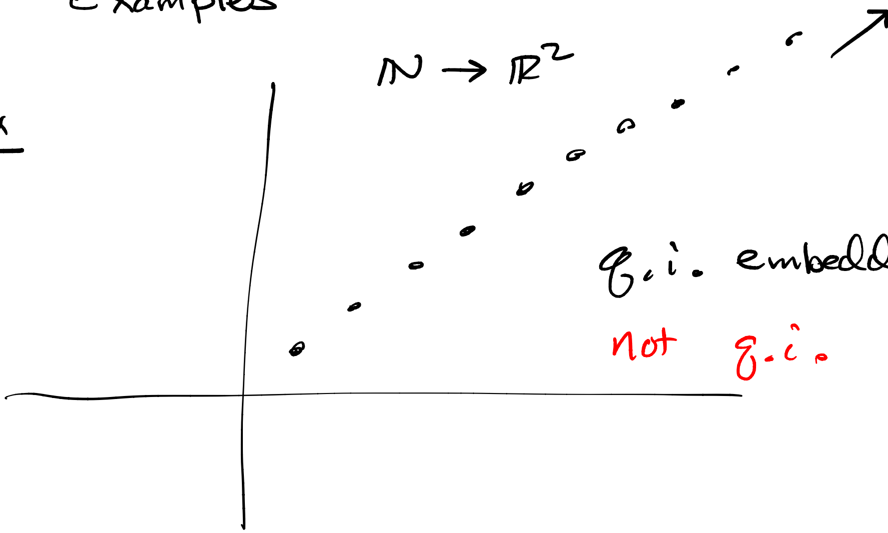
Ex $\exists$ q.i. $\mathbb{R}\to\mathbb{Z}$

What about $\mathbb{R}\to\mathbb{N}$? or $\mathbb{Z}\to\mathbb{N}$? NO

IV Examples

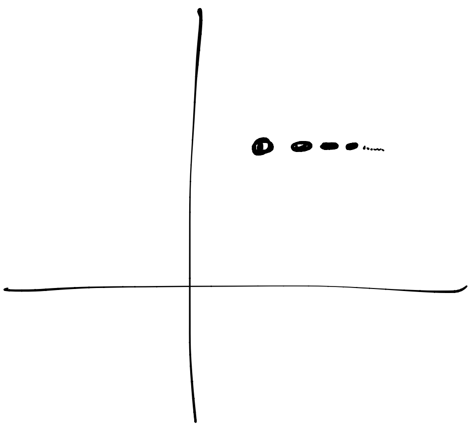
Ex

$\mathbb{N} \to \mathbb{R}^2$

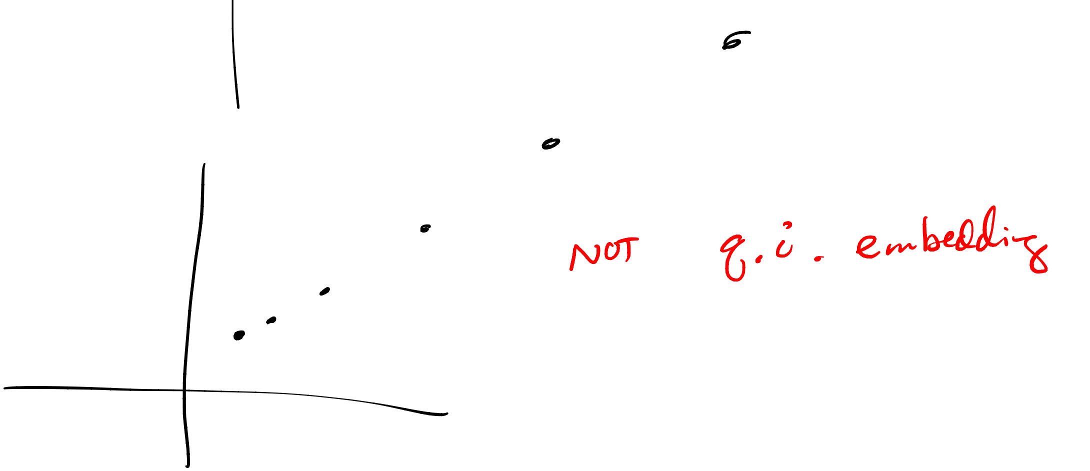


q.i. embedding

not q.i.

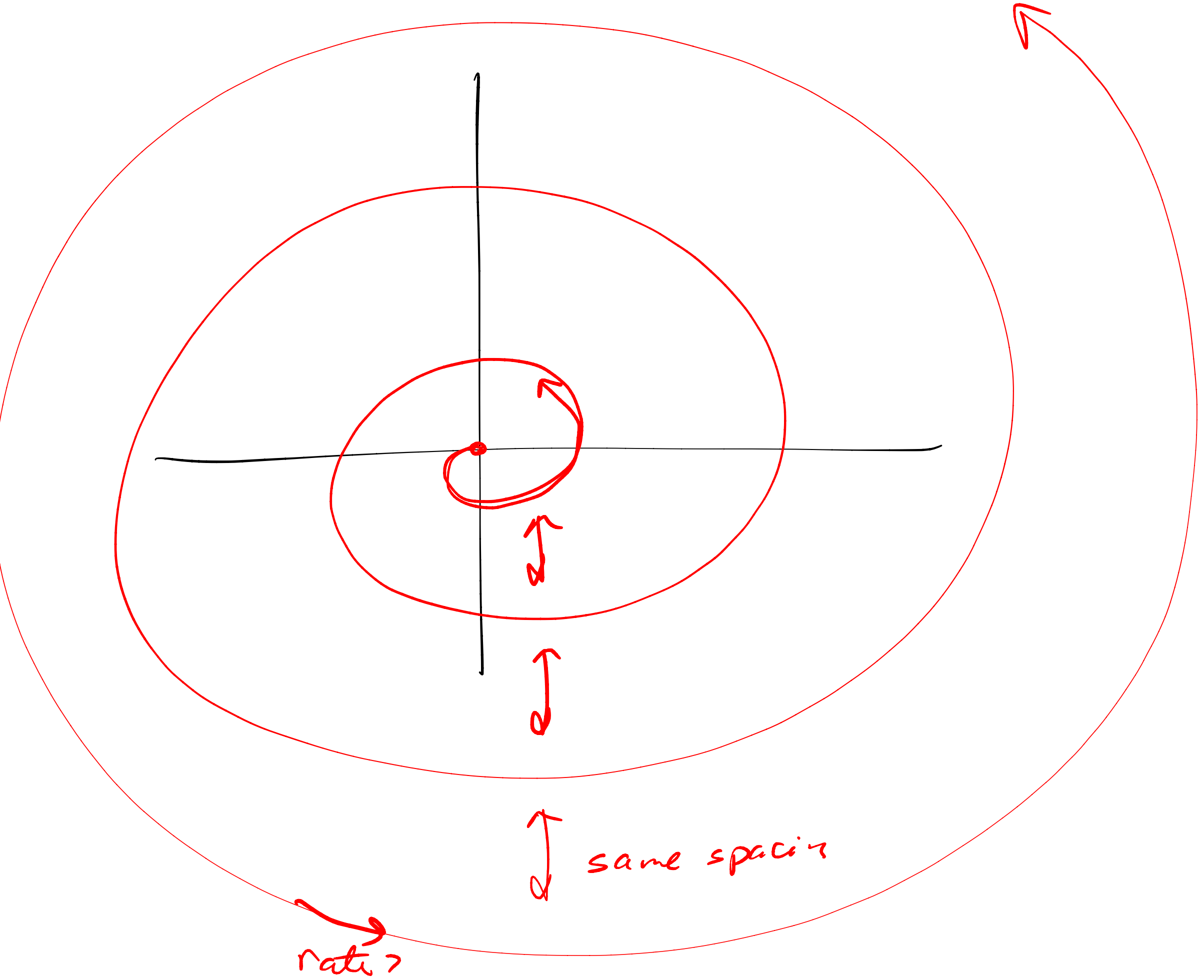


NOT q.i. embedding



NOT q.i. embedding

$[0, \infty) \to \mathbb{R}^2$



IT DEPENDS

(V) Q.I. IS AN EQUIVALENCE RELATION

& THERE IS A Q.I. GROUP

Thm Any two Cayley graphs for a group G w.r.t. finite gen sets are q.i.

$\Rightarrow$ # ends

δ-hyperbolicity

"type" of Dehn function

$\left.\begin{array}{l}\text{\# ends}\\ \delta\text{-hyperbolicity}\\ \text{"type" of Dehn function}\end{array}\right\}$ invariant under q.i.

"Coarse Geometry"

VI The Milnor-Schwarz Lemma
(FT OF GGT)

Def (X, d) IS **PROPER**

IS **GEODESIC**

$G \curvearrowright X$ IS **PROPERLY DISCONTINUOUS**

IS **COCOMPACT**

IS **ISOMETRIC**

Švarc

Lemma (Milnor - Schwarz)

If G is a group that acts on a proper, geodesic metric space (X, d) properly discontinuously cocompactly, and by isometries then G is finitely generated and

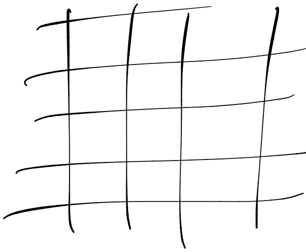
G is q.i. to X

every Cayley graph for G

Examples (non-Cayley Graph)

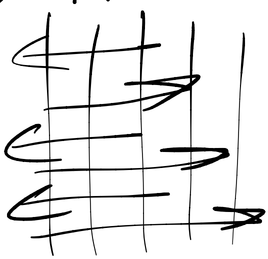
① $\mathbb{Z}^2$ & $\mathbb{Z}^3$

$\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ by integer translations



$\mathbb{Z}^2 \underset{q.i}{\cong} (\mathbb{R}^2, d_{eucl})$

② Klein Bottle Group K



$K \underset{q.i.}{\cong} (\mathbb{R}^2, d_{eucl.}$

③ If $[G:H] < \infty$ and G, H f.g.

then $G \underset{q.i}{\cong} H$

Proof

(I) Choose the ball & Define the Generating set.

(II) Repeat Steve's Proof

III Keep track of distances

(a) $c = \inf \{ d(B, gB) \mid g \in G \setminus \mathbb{1}, \ g \notin S \}$

(b) <u>Claim</u> $\forall \ g \in G \setminus (S \cup \{\mathbb{1}\})$

$$d(x_0, g x_0) \geq 2R + c$$

$\Rightarrow \quad \exists \ K \geq 2$ s.t.

$$R + (K-1) c \leq d(x_0, g x_0) \leq R + K c$$

The Q. I. part

Def $\varphi : G \to X$ by $\varphi(g) = g x_0$

(A) Coarsely surjective

(B) Q.I. embedding

- Simplify!

Let $L = \max\{ d(x_0, s x_0) \mid s \in S \}$

$K = \max(\frac{1}{c}, L, 2R)$

$C = \max(1/K, c)$

Case 1: $g = \mathbb{1}$

Case 2: $g \in S$

Case 3: $g \notin S \cup \{\mathbb{1}\}$