HW 9: The end(s) of groups

(1) Suppose that Γ is a connected graph. Make Γ into a metric space by declaring each edge to have length 1. Find an example of a graph G having a vertex v of infinite degree such that the closed ball $\{y \in G : d(v, y) \leq 1\}$ is not compact. A graph where every vertex has finite degree is called **locally finite**.

(2) (This problem is a little technical, but we'll use it in class.) Suppose that e is an end of a path connected metric space (X, d). Assume that X has the property that for every $x \in X$ and $\rho > 0$, the ball $\{y \in X : d(x, y) \leq \rho\}$ is compact (this is the case for $\mathbb{R}^n$ and for graphs where every vertex has finite degree.) . You may also use the fact that a compact set is closed and bounded. Let $x \in X$. Prove there is a ray r representing the end such that $\rho(0) = x$ and for every $N \in \mathbb{N}$, $d(x, r(t)) > N$ whenever $t > N$.

(3) (BONUS) We can "topologize" the set of ends $Ends(X)$ of a path-connected space X, by declaring $U \subset Ends(X)$ to be **open** if there exists a compact set $K \subset X$ and an unbounded set $Y \subset X \setminus K$ which is the union of path components, such that $r \in U$, if and only if $r(t) \in Y$ for $t >> 0$.

   (a) Show that if $U_\alpha \subset Ends(X)$ is open for all α in some index set, then $\bigcup_\alpha U_\alpha$ is open. (Hint: the intersection of compact sets is compact.)

   (b) Show that if $U_1, U_2 \subset Ends(X)$ are open, then so is $U_1 \cup U_2$. (Hint: the union of two compact sets is compact.)

   (c) Explain why both $Ends(X)$ and $\varnothing$ are open subsets of $Ends(X)$.

   The previous problems show that the set of open subsets form what is called a "topology" on $Ends(X)$. A function $f\colon Ends(X) \to Ends(X)$ is defined to be **continuous** if for every open $U \subset Ends(X)$, the inverse image $f^{-1}(U)$ is open. A **homeomorphism** is a continuous bijection with continuous inverse.

(4) Suppose that a group G acts on a path-connected metric space (X, d) via homeomorphisms. Show that G also acts on the set of ends of X. (BONUS: Show the action on $Ends(X)$ is also by homeomorphisms.)

(5) Suppose that G acts on a locally finite connected graph Γ. Let $H = \{h \in G : \forall e \in Ends(X), h \cdot e = e\}$.

   (a) Prove that H is a subgroup of G. (It is the subgroup that "acts trivially on ends")

   (b) Prove that $[G : H] \leq |\text{Perm}(Ends(\Gamma))|$. (That is: the index of H in G is at most the cardinality of the set of permutation of ends.) Hint: Think about the proof of the orbit-stabilizer theorem. You can't appeal directly to the theorem since G might be infinite.

   (c) Use the action of the free group of rank 2 on its Cayley-graph to show that we may not be able to get equality.

(6) Prove that if G and H are finitely generated infinite groups, then $G \times H$ has one end. (Remember: we sketched the proof in class.)