Topics in Algebra Scott Taylor

HW 6: Generators, Relations, Homomorphisms

This assignment gives you more practice with group presentations and group actions.

(1) In a group g, remember that the **commutator** of two elements a, b is $[a,b] = aba^{-1}b^{-1}$. Let $G = \langle S|R\rangle$ be given by a presentation. Let $R' = \{[s,t]|s,t \in S\}$ and let $G' = \langle S|R \cup R'\rangle$. Incidentally, $\langle\langle R'\rangle\rangle$ is called the **commutator subgroup** of G. Show the following:

(a) If there is a surjective homomorphism from G to a nonabelian group, then G is nonabelian.

(b) G' is abelian and there is a surjective homomorphism $\alpha\colon G \to G'$. (The group G' is called the **abelianization** of G)

(c) Whenever A is an abelian group such that there is a homomorphism $h\colon G \to A$, then there is a homomorphism $\overline{h}\colon G' \to A$ such that $h = \overline{h}\circ\alpha$. (That is, all homomorphisms to an abelian group factor through the abelianization of G.

(2) Let $G = \langle x, y|x^2y^{-3}\rangle$.

(a) Show that G also has presentation $\langle a, b|aba = bab\rangle$

(b) Construct as much of a Cayley graph for G as you have patience for.

(c) Show that the abelianization of G is isomorphic to $\mathbb{Z}$; in particular G is an infinite group.

(d) Find an element in G of infinite order.

(e) Find a surjection of G onto a nonabelian group. (Hint: try $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, but bonus points if you can find a finite one!) Conclude that G is a nonabelian infinite group whose abelianization is $\mathbb{Z}$.

(3) Suppose that $A = \langle S|R\rangle$ and $B = \langle T|U\rangle$. Also suppose that there is a subgroup $H < A$ and an injective homomorphism $\phi\colon H \to B$. Let $V = \{\phi(h)h^{-1} : h \in H\}$. Define $A *_H B = \langle S \cup T|R \cup U \cup V\rangle$. This group is called the **free product of A and B amalgamated along H**. The notation is misleading because the group depends on the map ϕ, not just the subgroup H.

(a) Let $A = \mathbb{Z}$ and $B = \mathbb{Z}$. Find a subgroup $H < A$ and a homomorphism $H \to B$ so that that the group G from the previous problem is isomorphic to $\mathbb{Z} *_H \mathbb{Z}$.

(b) Explain why the case when $H = \{\mathbb{1}\}$ produces the free product $A * B$.

(4) (Bonus) This problem is an important result in group theory. Working on it will help you internalize several of the methods we've covered so far. It proves a partial converse to Meier 3.28. It shows that in a fairly general setting, if a group acts on a graph then it is an amalgamated product. It can be generalized somewhat to weaken the hypotheses, but we won't explore that in our course. The branch of GGT that is concerned with this theorem and its ilk is called "Bass-Serre Theory" after the two mathematicians who pioneered it.

Let Γ be a bipartite graph and suppose that G is a group acting on it and that no element of G has order 2 (in particular, no element flips an edge of Γ). Suppose also that the action is edge transitive (i.e. there is a single orbit of edges) and that the action is transitive on each partite set (i.e. if the vertices are black and white, all black vertices are in the same orbit and all white vertices are in the same orbit.)

(a) Pick an edge e with black endpoint v and white endpoint w. Let $A_G = stab_G(v)$, $B_G = stab_G(w)$ and $H_G = stab_G(e)$. Explain why H_G is a subgroup of both A_G and B_G.

(b) Suppose that v' is a black vertex. Prove that the subgroup $stab_G(v')$ is a conjugate of $stab_G(v)$. A similar statement holds for the white vertices and for the edges.

(c) Explain why e is a fundamental domain for the action and explain what generating set it produces (as in Meier Thm. 1.55). Can you see why this generating set is potentially much larger than necessary?

(d) Let S_G be a generating set for A_G defined by taking every element of H_G as a generator, together with one element from each coset in A_G/H_G. Let S be a set disjoint from G but in bijection with S_G. Explain why there is a set of relations $R \subset S^*$ such that $A = \langle S|R\rangle$ is a presentation for A_G. Similarly, do the same thing for B_G using a generating set T_G so that B_G has presentation $B = \langle T|U\rangle$.

(e) From the definition of presentation, observe that while A and A_G are isomorphic groups, they are not "equal." In particular, we may assume that $S \cup S^{-1}$ and $T \cup T^{-1}$ are disjoint. Let $\phi_A\colon A \to A_G$ and $\phi_B\colon B \to B_G$ be the group isomorphisms (taking S to S_G, etc.). Let $H = \phi_A^{-1}(H_G)$ be the preimage of H_G. Define $\phi\colon H \to B$ by $\phi = \phi_B^{-1} \circ \phi_A$. Explain why ϕ is a well-defined group homomorphism, so that we may consider the group $G' = A *_H B$.

(f) Define $\psi\colon G' \to G$ by taking S_G to S and T_G to T and then extending over words. Prove that ψ is a well-defined group isomorphism. (You do not have to give full details.)