Topics in Algebra Scott Taylor

HW 4: Practice with Generators and Relations

Read Scott's notes on finding a presentation for D_∞.

The purpose of this assignment is to have you do something similar with a group that shows up in topology. In order to emphasize the techniques of this course, we won't explain where this group comes from right now.

Define functions $A, B\colon \mathbb{R}^2 \to \mathbb{R}^2$ by:

$$\begin{aligned} A(x,y) &= (x+1, y) \\ B(x,y) &= (-x, y+1). \end{aligned}$$

Notice that both A and B are isometries of the Euclidean plane. Let K be the group generated by A and B. That is K consists of all possible compositions of a finite number of compositions of A, B, A^{-1}, and B^{-1}. Observe that K is an infinite group since $A^n(x,y) = (x+n, y)$ for all $n \in \mathbb{Z}$, so A has infinite order. For $f, g \in K$, we let $f \cdot g = g \circ f$, and consider $\cdot$ as the group operation on K. (In other words, we reverse the order in which we write function composition.)

(1) In the paragraph above, we showed that A has infinite order. Now show that B also has infinite order. In the process, work out a formula for B^n for any $n \in \mathbb{Z}$.

(2) Explain why $S' = \{A \cdot B, B, B^{-1} \cdot A^{-1}, B^{-1}\}$ is also a generating set for K. This generating set will be used in a few questions to make life a bit easier. Set $C = A \cdot B$.

(3) Let Γ be the integer lattice in $\mathbb{R}^2$, having vertices consisting of all points $(a,b) \in \mathbb{Z}^2$ and edges of the form (a,b) with either a or b an integer. Notice that K acts of Γ.

  (a) Find a fundamental domain for the action of K on Γ.

  (b) What generating set for K does your fundamental domain produce (as in Meier Theorem 1.55)?

(4) Use the "Drawing Trick" described by Meier (Remark 1.49, page 25) to draw a Cayley graph for K using generating set $S = \{A^{\pm 1}, B^{\pm 1}\}$.

(5) Use your Cayley Graph from the previous problem to guess some relations (expressed in terms of the generating set $\{A^{\pm 1}, B^{\pm 1}\}$).

(6) Let $G' = \langle c, b | c^2 b^{-2} \rangle$. Note that the equation $c^2 b^{-2} = \mathbb{1}$ can be rewritten as $c^2 = b^2$.

Show that every element of G' can be written in one of the following forms:

$$c^{n_1} b^{\pm 1} c^{n_2} b^{\pm 1} c^{n_3} b^{\pm 1} \cdots c^{n_k}$$

where n_1 or n_k can be 0, but all other n_i are non-zero integers. This expression is unique but you don't have to prove it (even though you could).

(7) Explain why $C^2 \cdot B^{-2} = \mathbb{1}$ in G.

(8) Define $\psi\colon K \to G'$ as follows. Let $\psi(C) = c$ and $\psi(B) = b$ and extend to be a homomorphism using the fact that $\{C^{\pm 1}, B^{\pm 1}\}$ is a generating set for K. Show that K has presentation $\langle c, b | c^2 b^{-2} \rangle$.

(9) Briefly explain why $\langle a, b | abab^{-1} \rangle$ is also a presentation for G'. (Hint: set $c = ab$)