HW 2 #2

The infinite dihedral group. D_∞

Consider $D_\infty < \mathrm{Isom}(\mathbb{E}^1)$ to be the isometries of $\mathbb{R}$ (w/ eucl. metric) that take $\mathbb{Z}$ to $\mathbb{Z}$.

let $A(x) = -x$ and $B(x) = 1-x$.

A is reflection of $\mathbb{R}$ across 0 and

$B(x)$ is reflection across $1/2$.

proof: $B(1/2) = 1/2$ but $B \neq id$.

In fact $B(x) - 1/2 = 1-x - 1/2 = \frac{1}{2} - x$

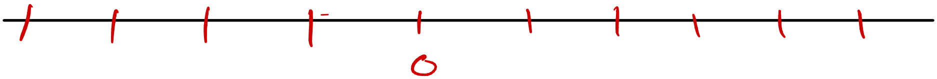
so $B(x)$ is the same distance from $1/2$ as x

but on the opposite side. $\square$

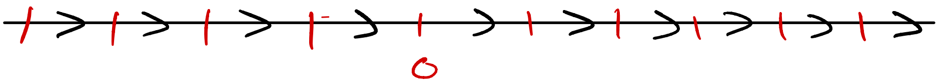
We claim that A and B generate D_∞.

proof on next page.

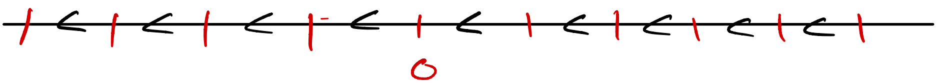
Consider $\mathbb{R}$ with $\mathbb{Z}$ marked.



$\mathbb{R}$ can be oriented left to right



or right to left



Each isometry of $\mathbb{R}$ either preserves orientation or reverses orientation and is completely determined by whether it is orientation-preserving or orientation-reversing, and the value of $T(0)$.

(proof Given $T(0)$, there are two choices for $T(1)$:
$T(0)+1$ or $T(0)-1$, if the former then
$T(n) = T(0)+n \quad \forall n \in \mathbb{Z}$, if the latter $T(0)-n$ then)

Both A and B are reflections, so they reverse orientation. Thus, $(A \circ B)^n$ and $(B \circ A)^n$ preserve orientation, for all n.

Notice
$$A \circ B\,(x) = -(1-x) = x-1$$
$$B \circ A\,(x) = 1-(-x) = x+1 \qquad \forall x \in \mathbb{R}$$

Let $T \in D_\infty$.

Case 1 If T is orientation preserving and $T(0) \geq 0$ then

$$(B \circ A)^{T(0)}(0) = 0 + \underbrace{1 + 1 + \cdots + 1}_{T(0)} = T(0)$$

Since $(B \circ A)^{T(0)}$ and T are both orientation-preserving and take 0 to the same location, $T = (B \circ A)^{T(0)}$

Case 2 T is orientation-preserving and $T(0) \leq 0$

As in Case 1, $T = (A \circ B)^{|T(0)|}$.

Case 3 T is orientation-reversing and $T(0) > 0$ then $T \circ A$ is orientation-reversing and $T \circ A(0) < 0$. By Case 2 $T \circ A = (AB)^n$ for some $n \in \mathbb{N}$ so $T \circ \underbrace{A \circ A}_{1} = (A \circ B)^n \circ A$

Case 4 T is orientation-reversing and $T(0) < 0$

As in Case 3, $T \circ B = (B \circ A)^n$ for some n

so $T = (B \circ A)^n \circ B$. $\blacksquare$