Topics in Algebra Scott Taylor

HW 2

Reread Clay-Margalit Sections 1.2, 1.3.

(1) Suppose that G is a group and that $H < G$ is a subgroup of index 2 (i.e. $|G/H| = 2$). Remember that this means that there is a $g \in H$ such that for each $g' \in G$ there exists $h \in H$ such that either $g' = h$ or $g' = gh$. Prove that H is normal in two ways:

(a) First just use the definition of "normal": Suppose $g \notin H$ and prove $gHg^{-1} = H$ by contradiction, using the fact that the two cosets are H and gH.

(b) Define $\phi\colon G \to \mathbb{Z}/2\mathbb{Z}$ by $\phi(g) = \begin{cases} 0 & g \in H \\ 1 & g \notin H \end{cases}$. Prove that ϕ is a homomorphism and then use the fact that kernels of homomorphisms are normal subgroups.

(2) (Try this problem first without reading about it in the textbooks. If you get stuck you can look at the textbooks for it. If you do, be sure to say so.) Let D_∞ be the group whose elements are isometries T of $\mathbb{R}$ with the property that $T(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$. (i.e. "integer-preserving isometries of the reals"). The group operation is function composition. This is called the "infinite dihedral group." Let $A(x) = -x$ for every $x \in \mathbb{R}$ and let $B(x) = 1 - x$.

(a) Show that $B\colon \mathbb{R} \to \mathbb{R}$ is the reflection of $\mathbb{R}$ across the number $1/2$.

(b) Show that A and B generate D_∞

(c) Explain why for every $g \in D_\infty$ there exists $n \in \mathbb{N} \cup \{0\}$ and $\epsilon \in \{0, 1\}$ such that either

- $g = (AB)^n A^\epsilon$, or
- $g = (BA)^n B^\epsilon$

(3) Consider the symmetric group S_4 i.e. the permutation group of 4 points.

(a) Give an example of a nontrivial, proper normal subgroup of S_4.

(b) Give an example of a nontrivial proper non-normal subgroup of S_4.

(c) Choose one of your examples (call it H). Find a generating set for H and then use cosets to extend it to a generating set for S_4.