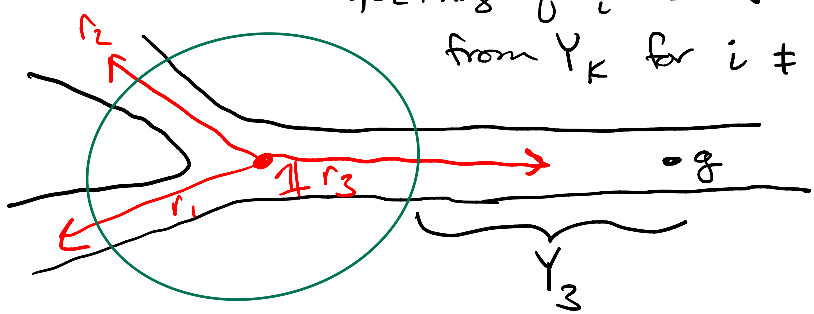
Theorem (Freudenthal–Hopf)

A f.g. group has 0, 1, 2, or ∞ly many ends.

proof Suppose that G is a group with at least $K \geq 3$ ends. We will show it has at least $2K-2$ ends. (And thus it must have ∞ly many ends.) Let Γ be a Cayley graph for G with respect to a finite generating set S.

Let $r_1, \dots, r_K$ be rays based at $\mathbb{1} \in \Gamma$ representing distinct ends. By definition, $\exists\, \rho > 0$ s.t. $r_1(t), r_2(t), \dots, r_K(t)$ lie in distinct components of $\Gamma \setminus \underbrace{B(\mathbb{1}, \rho)}_{\text{closed ball of radius } \rho \text{ centered at } \mathbb{1}}$ for $t >> 0$. Let Y_K be the component containing $r_K(t)$ for $t >> 0$. It is unbounded since r_K is a ray. We may assume that the image of r_i is disjoint from Y_K for $i \neq K$.

Ex
$K=3$

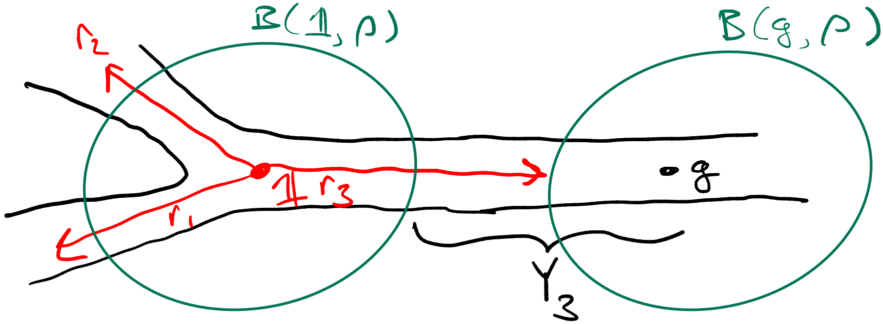


let $g \in Y_K$ be such that $d(\mathbb{1}, g) > 2\rho$

Notice that $g \cdot B(\mathbb{1}, \rho) = B(g, \rho)$ and

$$B(\mathbb{1}, \rho) \cap B(g, \rho) = \emptyset$$

$\underline{Ex}$
$K=3$



The rays $r_1, \dots, r_{K-1}$ all lie in a single unbounded component of $\Gamma \setminus B(g, \rho)$ as they are disjoint from Y_K and all start at $\mathbb{1}$. However the action of g on Γ is an isometry so $\Gamma \setminus g \cdot B(\mathbb{1}, \rho)$ $= \Gamma \setminus B(g, \rho)$ has at least K unbounded components. One of them contains $r_1, \dots, r_{K-1}$. There are rays $r_1', \dots, r_{K-1}'$ in each of the other components based at g. Those rays are all not equivalent to any of $r_1, \dots, r_{K-1}$ so there are at least $2(K-1)$ ends of Γ. See the next page for a picture.

(the rays r_1' and r_2')