Free products and Actions on Trees.

let A, B be groups w/ presentations

$\langle S \mid R \rangle = A \qquad \langle T \mid \mathcal{U} \rangle = B.$

<u>Def 1</u> $\quad A * B = \langle S \cup T \mid R \cup \mathcal{U} \rangle$

<u>Def 2</u> $\quad A * B = \left\{ \begin{array}{l} \text{words of the form} \\ a_1 b_1 a_2 b_2 \cdots a_n b_n \\ \text{w/ } a_i \neq 1 \quad i \neq 1 \\ \quad\; b_i \neq 1 \quad i \neq n \\ a_i \in A,\ b_i \in B \end{array} \right\}$

(These definitions are equivalent)

<u>Thm 1</u> $\forall A, B$ $\exists$ tree T s.t. $A * B$ acts on T with $\mathrm{stab}(e) = 1$ $\forall$ edges $e \in E$ & action is non-trivial

<u>Thm 2</u> If a group G acts non-trivially on a tree T s.t.

(1) $\mathrm{stab}(e) = 1$ $\forall$ edges e

(2) the action is transitive on edges

(3) T is bipartite & action preserves the partition

then $\exists$ non-trivial groups A, B. s.t

$$G \cong A * B$$

Proof of Theorem 1

Let $\mathcal{A} = \underbrace{(A*B)/A}_{\text{cosets of } A}$ $\quad \mathcal{B} = \underbrace{(A*B)/B}_{\text{cosets of } B}$

Let $\mathcal{A} \cup \mathcal{B}$ be vertices.

for $gA \in \mathcal{A}$ and $g'B \in \mathcal{B}$

put an edge joining gA to $g'B$ whenever $g = g'$.

gA —— gB

Let T be the resulting graph

Ex If $a \in A$ $b \in B$ are non-trivial then:

aB —— A —— B —— bA is a part of T

Claim 1 $A*B$ acts on T

Pf it is just the usual left action.

If $u \in A*B$ then $u\left(gA \text{ —— } gB\right) = ugA$ —— ugB

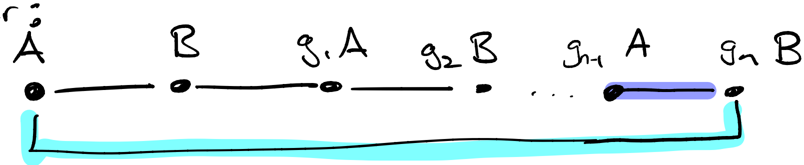
$\square$

Claim 2 T does not have a cycle.

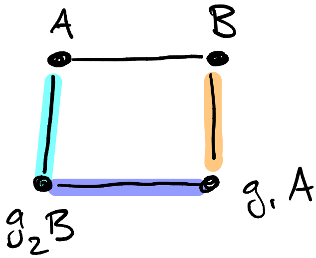
Suppose g_1A —— g_2B —— g_3A —— g_4B $\cdots$ g_5B is a cycle

Then act by g_1^{-1} to get a cycle based at A

we renumber:

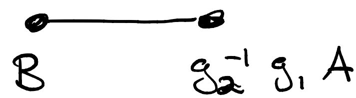


For simplicity, assume $n = 2$. General case is similar



Look at edge. By defn $g_2 \in A$

look at edge. We have



so $g_2^{-1} g_1 \in B \Rightarrow$

Look at edge: Notice $g_1 \in B \Rightarrow$

Thus, $\underbrace{g_2}_{\in A} \underbrace{(g_2^{-1} g_1)}_{\in B} = \underbrace{g_1}_{\in B}$

has two reduced words representing same element which contradicts Defn 2. $\square$

__Claim 3__ T is connected

__pf__ let $g \in G$ and consider gA

We can write $g = a_1 b_1 a_2 b_2 \cdots a_n b_n$ as in Def 2.

then notice:

$$gA \;\text{—}\; g b_n^{-1} B \;\text{—}\; g b_n^{-1} a_n^{-1} A \;\text{—}\; g b_n^{-1} a_n^{-1} b_{n-1}^{-1} B \;\text{—}\; \cdots \;\text{—}\; A$$

$$g b_n^{-1} B = gB, \qquad g b_n^{-1} a_n^{-1} A = g b_n^{-1} A$$

is a path from gA to A. Since $gA \;\text{—}\; gB$

all vertices are joined to A or B by a path.

so T is connected.

Claims 1 & 2 $\Rightarrow$ T is a tree.

Action is non trivial b/c $bA \neq B$ for $b \in B$

so $b \notin \text{stab}(\bullet_A)$

If e is an edge $gA \overset{e}{\text{—}} gB$ and $u \in \text{stab}(e)$

then $ugA = gA$ and $ugB = gB$ (u can't flip b/c cosets are disjt)

$\Rightarrow$ $u \in gA \cap gB$

$\Rightarrow$ $A \cap B \neq \emptyset \Rightarrow \otimes$. $\square$ (Thm 1)

Proof of Thm 2

Suppose G acts on a tree T as described, let e be an edge of T

let $A = \mathrm{stab}(\bullet)$ and $B = \mathrm{stab}(\circ)$

we claim $G \cong A * B$.

Claim 1 The stabilizer of any black vertex is a conjugate of A. The stabilizer of any white vertex is a conjugate of B.

pf. let v be a black vertex that is an endpt of an edge e'. Since the action is transitive on edges $\exists\, g \in G$ s.t. $g(e') = e$. Since the action preserves the partite sets $g(v) = \bullet$. Define $f: \mathrm{stab}(v) \to \mathrm{stab}(\bullet)$ by

$$f(h) = ghg^{-1}$$ Note $ghg^{-1}(\bullet) = \bullet$

Hence $g\,\mathrm{stab}(v)\,g^{-1} \subset \mathrm{stab}(\bullet)$. If $h' \in \mathrm{stab}(\bullet)$ then $g^{-1}h'g \in \mathrm{stab}(v)$ and $f(g^{-1}h'g) = h'$ so f is onto. Thus $\mathrm{stab}(\bullet) = g\,\mathrm{stab}(v)\,g^{-1}$ $\square$

We now define $\Phi : A * B \to G$

by extending the identity maps $\iota_A : A \hookrightarrow G$

$\iota_B : B \hookrightarrow G$

Across $A * B$ (i.e. recall A, B are subgroups of G)

Ex $\Phi(\underbrace{a_1 b_1 a_2 b_2 \cdots a_n b_n}_{\substack{\text{abstract word} \\ \text{using concatenation}}}) = \underbrace{a_1 b_1 \cdots a_n b_n}_{\substack{\text{element of } G \\ \text{(using } G\text{'s operation)}}}$

Clearly Φ is a homomorphism.

Claim 2 Φ is injective.

Let $a_1 b_1 a_2 b_2 \cdots a_n b_n$ be a reduced word in $A * B$

we show it is not the identity in G.

For simplicity assume $a_1 \neq 1\!\!1$, $b_n \neq 1\!\!1$

$$a_1 b_1 a_2 b_2 \cdots a_n b_n = \underbrace{(a_1 b_1 a_1^{-1})}_{\text{stabilizes a white vertex}} \underbrace{(a_1 a_2) b_2 (a_1 a_2)^{-1}}_{\text{stabilizes a white vertex}}$$

$$(a_1 a_2 a_3) b_3 (a_1 a_2 a_3)^{-1} a_4 b_4 \cdots$$

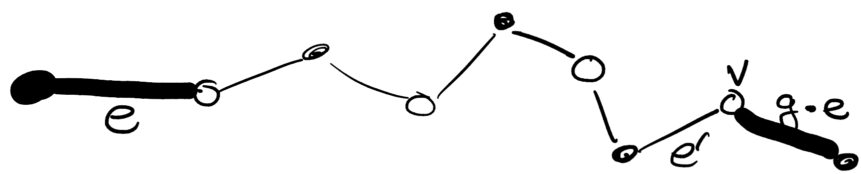
$$\cdots b_n$$

is a product of conjugates of elements of B, times an element of A. This can be used to show it isn't the identity, similar to Ping-pong lemma

Claim 3 Φ is surjective

It suffices to show $A \cup B$ generates G.

Let $g \in G$. Then $g \cdot e \neq e$



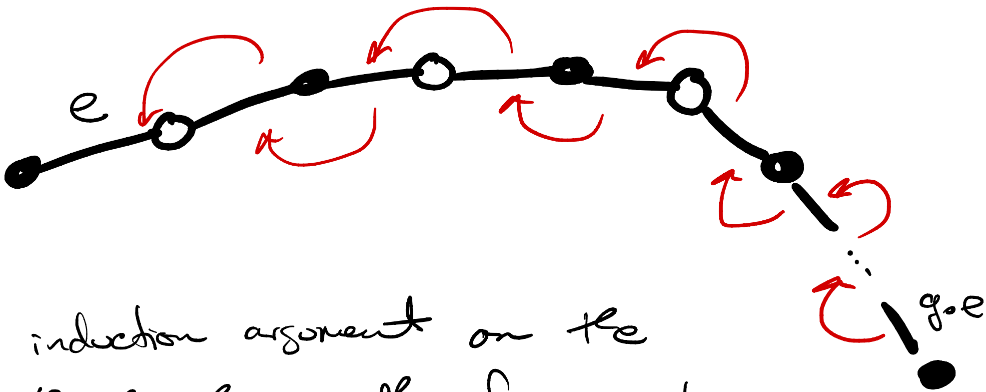
Choose a path from e to $g \cdot e$

WLOG assume path ends at a white vertex v.

Notice $g^{-1} B g = \text{stab}(v)$ let e' be the previous edge. The action is transitive on edges so $\exists\, g'$ s.t. $g' \cdot e = e'$

$\Rightarrow (g' g^{-1}) \cdot (g \cdot e) = e'$ and preserves black & white vertices. Thus, $g' g^{-1} \in \text{stab}(v)$

So we get a sequence of group elements where each one is in the stabilizer of the "next" vertex along the path & whose product is G



An induction argument on the length of the path from e to g·e proves the result. For

$$g' g^{-1} \in \mathrm{stab}(v) \in g \underbrace{\mathrm{stab}(o)}_{B} g^{-1}$$

$\Rightarrow \exists\, b \in B$ s.t.

$$g' = g\, b$$

$\Rightarrow$

$$g = g' b^{-1}$$

product of elements from A, B by induction.

$\square$

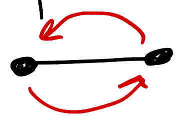
__Lemma__ Suppose a finite group G acts on a tree $\mathcal{T}$. Then $\exists\, p \in \mathcal{T}$ s.t.

$$g \cdot p = p \quad \text{for all } g \in G$$

__pf__ Choose $x \in \mathcal{T}$. Then $\mathrm{orb}_G(x)$ is finite. Let $\mathcal{T}_0$ be the smallest subtree of $\mathcal{T}$ containing $\mathrm{orb}_G(x)$.

If $\mathcal{T}_0$ has one vertex, let $p = \mathcal{T}_0$

if $\mathcal{T}_0$ has two vertices,

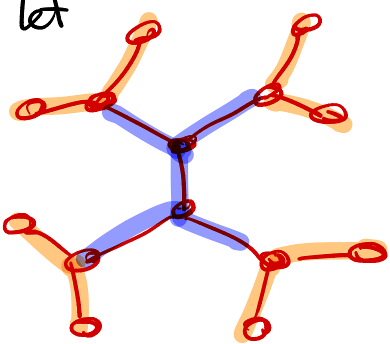


let p = center of $\mathcal{T}_0$

if $\mathcal{T}_0$ has > 2 vertices let

$\mathcal{T}_1 = \mathcal{T}_0 \setminus (\text{leaves of } \mathcal{T}_0)$

Note G acts on $\mathcal{T}_1$



Continuing we arrive at a base case and so prove the lemma. $\square$

Cor Every finite subgroup of $A * B$ is conjugate to a subgroup of A or a subgroup of B

proof Let $G < A * B$ be finite.

$A * B$ acts on a tree with trivial edge stabilizers by Theorem 1.

By the lemma G fixes a point.

So $\exists$ vertex v with $G < \mathrm{stab}(v)$

But $\mathrm{stab}(v)$ is a conjugate of A or B.

$\square$