A presentation for D_∞

Let $A = \{a, b\}$ and $S = \{a, b, a^{-1}, b^{-1}\}$.

Let F be the free group $F(S)$ consisting of words in the letters a, b, a^{-1}, b^{-1} up to the cancellation and insertion of pairs of inverse letters. For ex. $a\,ba^{-1}a\,b = ab^2$.

Let $A : \mathbb{R} \to \mathbb{R}$ be the function $A(x) = -x$ and $B : \mathbb{R} \to \mathbb{R}$ be the function $B(x) = 1 - x$

The group D_∞ is defined to be the group of isometries of $\mathbb{R}$ taking $\mathbb{Z}$ to $\mathbb{Z}$. We saw before that D is generated by A and B.

For the purposes of this problem, let $\cdot$ denote the group operation on D_∞, so

$$A \cdot B \quad \text{is the function} \quad B \circ A$$

first do A
then B.

Define $\varphi : F \to D_\infty$ by

$$\varphi(a) = A$$
$$\varphi(b) = B$$

and extending over words. For example

$$\varphi(ababb) = \varphi(a)\cdot\varphi(b)\cdot\varphi(a)\cdot\varphi(b)\cdot\varphi(b)$$
$$= A \cdot B \cdot A \cdot B \cdot B$$
$$= \underbrace{B \circ B \circ A \circ B \circ A}$$

function $\mathbb{R} \to \mathbb{R}$ using usual
right ~~to left~~ function composition

Observe that φ is (automatically) a homomorphism. It is surjective b/c A, B generate D_∞.

Let $R = \{a^2, b^2\} \subset F$.

We will show $F/\langle\langle R\rangle\rangle \cong D_\infty$

normal closure

Thm $F/\langle\langle R\rangle\rangle \cong D_\infty$

pf We construct an isomorphism directly.

For convenience, for $w \in F$, let $[w]$ denote its image in the quotient group $F/\langle\langle R\rangle\rangle$.

Define $\psi : D_\infty \to F/\langle\langle R\rangle\rangle$

as follows For $g \in D_\infty$ write it as

$(A\cdot B)^n \cdot A$ or $(B\cdot A)^n \cdot B$ or $(A\cdot B)^n$ or $(B\cdot A)^n$

for some $n \in \mathbb{Z}$, $n \geq 0$. Let

$$\psi\left((A\cdot B)^n \cdot A\right) = ([a][b])^n [a] \quad \text{etc.}$$

Since $A^2 = B^2 = \mathbb{1}$ it is easy to check this is a group homomorphism. It is injective b/c $(A\cdot B^n) = (B\circ A)^n$ is the function $x \mapsto x+n$ etc.

It remains to show it is surjective.

Let $[w] \in F/\langle\langle R\rangle\rangle$. We will show $w \in \mathrm{Im}(\psi)$.

Each group element of F, can be represented as a reduced word. We let w be a reduced word -- the class $[w]$.

We may write w in the generators as

$$w = a^{n_1} b^{m_1} a^{n_2} b^{m_2} \cdots a^{n_k} b^{m_k}$$

where $n_2, \ldots, n_k \in \mathbb{Z} \setminus \{0\}$

$m_1, \ldots, m_{k-1} \in \mathbb{Z} \setminus \{0\}$

Then

$$[w] = [a]^{n_1} [b]^{m_1} \cdots [a]^{n_k} [b]^{m_k}$$

Since $R = \{a^2, b^2\}$, $[a]^2 = [b]^2 = \mathbb{1}$

So we can repeatedly reduce exponents mod 2 until $[w]$ is of the form $([a][b])^n [a]$, $([a][b])^n$, $([b][a])^n$, or $([b][a])^n [b]$. That is the $[a]$ and $[b]$ alternate. Then

$\psi((AB)^n a) = ([a][b])^n [a] = [w]$ or

$\psi((AB)^n) = ([a][b])^n = [w]$, or

$\psi((BA)^n B) = ([b][a])^n [b] = [w]$, or

$\psi((BA)^n) = ([b][a])^n = [w]$

Thus ψ is surjective. $\square$