Class Work 8: Ends of Groups

This is to be completed together as a class, to establish shared notation and terminology. This is based on Theorem 8.32 from Bridson-Haefliger. Meier's text has a particularly simple proof, but there are interesting features of this one.

Definition 1. Suppose that (X, d) is a metric space. A subset $K \subset X$ is **compact** if every sequence (x_n) in K has a subsequence that converges to a point in K. The space (X, d) is **path-connected** if for all $a, b \in X$ there exists a continuous $\alpha \colon [0, 1] \to X$ such that $\alpha(0) = a$ and $\alpha(1) = b$. A function $f \colon X \to Y$ between metric spaces is **proper** if for every compact set $K \subset Y$, the set $f^{-1}(K)$ is also compact. A **ray** in X is a proper, continuous function $r \colon [0, \infty) \to X$. For rays r, r' we define $r \sim r'$ if for all compact $K \subset X$ and $t >> 0$, $r(t)$ and $r'(t)$ are in the same path component of $X \setminus K$. An equivalence class of rays is called an **end** of X.

Definition 2. Suppose that G is a finitely generate group and that Γ is a Cayley graph corresponding to a finite generating set. The **number of ends of** G is defined to be the number of ends of Γ. It will follow from our discussion of quasi-isometries that the choice of generating set does not matter (as long as it is finite).

Today's goal is to prove:

Theorem 3 (Freudenthal-Hopf)**.** *If G is a finitely generated group then it can have only 0, 1, 2 or infinitely many ends.*

Throughout let G be a finitely-generated group and let Γ be its Cayley graph with respect to a finite generating set S. Let $H < G$ be the subgroup of group elements that stabilize each ray. Recall that H is also a set of vertices in Γ. Assume that Γ has finitely many ends.

(1) Explain why $[G : H] < \infty$.

(2) Explain why there is a constant C (depending only on $[G : H]$) such that each vertex of Γ is within distance C of H.

(3) Explain why there is a ray r_0 such that:

   (a) $r_0(n) \in H$ for all $n \in \mathbb{N}$.

   (b) $r_0(0) = \mathbb{1}$

   (c) $d(r_0(n), \mathbb{1}) \geq n$ for all $n \in \mathbb{N}$.

   Let e_0 be the end represented by r_0. Let $\gamma_n = r_0(n)$.

(4) Suppose that r_1, r_2 are rays representing distinct ends $e_1, e_2 \neq e_0$. WLOG, $r_i(0) = \mathbb{1}$ and $d(r_i(n), \mathbb{1}) \geq n$ for all $n \in \mathbb{N}$ and $i \in \{1, 2\}$.

   Explain why there is a $\rho > 0$ such that $r_0(t), r_1(t), r_2(t)$ are all in different path components of $\Gamma \setminus B(\mathbb{1}, \rho)$. Also explain why this means that $d(r_1(t), r_2(t')) \geq 2\rho$ whenever $t, t' > 2\rho$.

(5) For $n > 3\rho$, explain why $\gamma_n \cdot r_i(0)$ lies in a different path component of $\Gamma \setminus B(\mathbb{1}, \rho)$ from $r_i([\rho, \infty))$.

(6) Explain why there exist $t, t' > 2\rho$ such that $\gamma_n \cdot r_1(t), \gamma_n \cdot r_2(t') \in B(\mathbb{1}, \rho)$.

(7) Find a contradiction and conclude that if Γ has 3 or more ends, then it has infinitely many.

Here are some additional points for discussion:

(1) Show that for any finite index subgroup H of G, the groups G and H have the same number of ends.

(2) Show that the set of ends of G is (sequentially) compact.

(3) Show that G has a finite index subgroup isomorphic to $\mathbb{Z}$ if and only if G is 2-ended.