Topics in Algebra Scott Taylor

Class Work 7: Ends of Spaces and Groups

This is to be completed together as a class, to establish shared notation and terminology. This is based on Office Hour 10 from Clay and Margalit.

Definition 1. Suppose that (X, d) is a metric space. A subset $K \subset X$ is **compact** if every sequence (x_n) in K has a subsequence that converges to a point in K. The space (X, d) is **path-connected** if for all $a, b \in X$ there exists a continuous $\alpha\colon [0, 1] \to X$ such that $\alpha(0) = a$ and $\alpha(1) = b$.

Definition 2. A function $f\colon X \to Y$ between metric spaces is **proper** if for every compact set $K \subset Y$, the set $f^{-1}(K)$ is also compact.

Definition 3. A **ray** in X is a proper, continuous function $r\colon [0, \infty) \to X$.

For rays r, r' we define $r \sim r'$ if for all compact $K \subset X$ and $t >> 0$, $r(t)$ and $r'(t)$ are in the same path component of $X \setminus K$.

Definition 4. An equivalence class of rays is called an **end** of X.

(1) For all $n \in \mathbb{N}$, find a metric space with n ends.

(2) Find a space with countably many ends.

(3) Find a space with uncountably many ends.

(4) Show that the number of ends of a space is equal to the supremum of the number of unbounded path components of $X \setminus K$, where the supremum is over all compact subsets $K \subset X$.

Definition 5. Suppose that G is a finitely generate group and that Γ is a Cayley graph corresponding to a finite generating set. The **number of ends of** G is defined to be the number of ends of Γ. It will follow from our discussion of quasi-isometries that the choice of generating set does not matter (as long as it is finite).

Lemma 6. *If G, H are finitely generated infinite groups, then $G \times H$ is one-ended.*

Theorem 7 (Freudenthal-Hopf)**.** *If G is a group then it can have only 0, 1, 2 or infinitely many ends.*