Class Work 6: The word problem

This is to be completed together as a class, to establish shared notation and terminology. This is based on Meier, Sections 5.3 and 5.4 as well as Office Hour 8 from Clay and Margalit.

1. Algorithms and Constructions

Given a group presentation $G = \langle S|R\rangle$. The **word problem** is to find an algorithm that can detect whether or not a given word ω in the generators and their inverses represents the identity in G. For our purposes, an "algorithm" is a finite list of instructions, hypothetically implementable in a computer, that can turn a given input into an output in a finite amount of time.

In a metric space (such as a graph) for a point x, the **(closed) ball** of radius x of radius r is:

$$\mathcal{B}(x,r) = \{y \in X : d(x,y) \leq r\}$$

The **sphere of radius** r is:

$$\mathcal{S}(x,r) = \{y \in X : d(x,y) = r\}.$$

We will often apply these notions to groups, in which case X will be the vertices of the Cayley graph and distance is the edge-path distance.

A Cayley graph Γ is **constructible** if there is an algorithm that for a given n lists all the vertices in $\mathcal{B}(\mathbb{1}, n)$ and all the edges between those vertices.

Theorem. *For a given group G with presentation $\langle S|R\rangle$, the word problem is solvable if and only if the Cayley graph is constructible.*

2. Baumslag-Solitar, again

Recall that $BS(1,n) = \langle a, b | ab = b^2 a\rangle$.

(1) What is a normal form for $BS(1,n)$?

(2) Construct the part of the Cayley graph corresponding to all words of the form $a^k b^\ell$ and $b^k a^\ell$ for $k, \ell \in \mathbb{Z}$.

(3) How should we think about adding in the words corresponding to aba, ab^3a, ab^5a etc.

(4) Come up with a general description of the Cayley graph?

3. Isoperimetric problems

Consider a group presentation $\langle S|R\rangle$ for a group G. A word ω that represents $\mathbb{1}$ can be converted to $\mathbb{1}$ by insertions, deletions of pairs aa^{-1} for $a \in S \cup S^{-1}$ and insertions, deletions of relators r and their cyclic rearrangments. The **Dehn function** is $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(n) = N$ that counts, for a word of length n, representing $\mathbb{1}$, the minimal number N insertions and deletions of relators and their cyclic rearrangments are needed. The minimum is taken over all such words and all such sequences.

(1) Compute the Dehn function for $\langle a|a^m\rangle$.

(2) Show that the Dehn function f of a finite group (with finite presentation) has $f(n) \leq Cn$ for some constant C.

(3) Show that the Dehn function f of $\mathbb{Z} \times \mathbb{Z}$ is bounded above by $f(n) \leq n^2$. What about other finitely generated abelian groups?

(4) Show that the word problem for $\mathbb{Z} \times \mathbb{Z}$ is solvable in linear time.

(5) Express the Dehn function in terms of an isoperimetry problem.