Class Work 5: Free Products and Surface Groups

This is to be completed together as a class, to establish shared notation and terminology. This is based on Meier, Sections 3.7 - 3.9.

1. Free products of finite groups are virtually free

Let A and B be nontrivial groups.

(1) Review the definition of $A * B$ and explain why $A * B = B * A$.

(2) Explain why $\iota_A \colon A \to A*B$ defined by $\iota_A(a) = a$ for all $a \in A$ is an injective homomorphism (and why this is even something that has to be proven!) We identify $A = \iota_A(A)$. Similarly, $B = \iota_B(B)$.

(3) Let G be a group and suppose that $\phi_A \colon A \to G$ and $\phi_B \colon B \to G$ are homomorphisms. Prove that there is a unique homomorphism $\phi \colon A * B \to G$ such that for each $a \in A, b \in B$, $\phi(a) = \phi_A(a)$ and $\phi(b) = \phi_B(b)$ for all $a \in A, b \in B$.

(4) Let $\phi \colon A * B \to A \oplus B$ be the extension of the coordinate maps $A \to A \oplus B$ and $B \to A \oplus B$. Explain why $K = \ker \phi$ is generated by the commutators: $S = \{[a, b] : a \in A \setminus \mathbb{1}, b \in B \setminus \mathbb{1}\}$. If A and B are finite, show that $[G : K] = |A||B|$.

(5) We claim that K is a free group on S. Describe how an algebraic argument of this fact would look.

(6) Review the construction of a nontrivial action of $A * B$ on a tree $\mathcal{T}$. What are the stabilizers of the edges and vertices? Use this to show that K acts freely on $\mathcal{T}$ and, therefore, must be a free group. What is its rank?

(7) Suppose that $H < A * B$ is finite. Prove that there exists $g \in A * B$ such that $g^{-1}Hg$ is a subgroup of either A or B. What does this imply about elements of finite order?

2. Surface Groups

The surface group S_g is defined by:

$$S_g = \langle a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g | [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle$$

(1) Prove that if $S_g \cong S_{g'}$ then $g = g'$

(2) Prove that S_g is abelian for $g = 1$ and nonabelian for $g \geq 2$.

(3) Do your best to construct a Cayley graph for S_g for $g = 1, 2$