Class Work 4: A subgroup of a free group

This is to be completed together as a class, to establish shared notation and terminology. This is based on Meier, Section 3.2 and Proposition 2.2.

1. An interesting subgroup of a free group

Definition 1.1. Suppose that F is a group and that $S \subset F$. We say that F is a **free group on** S (or with **basis** S) if whenever w is a reduced word in the letters $S \cup S^{-1}$, w does not represent the trivial element of F. For a given basis S, the **length** $|w|$ of a group element $w \in F$, is the number of letters in a reduced word representing w.

Let F be the free group on $\{x, y\}$. Set $a = x^2$, $b = xy$, and $c = xy^{-1}$. Let $H < F$ be the subgroup generated by a, b, c. The purpose of this discussion is to show:

Theorem. *H is a free group on $\{a, b, c\}$. In particular, a free group of rank 2, contains a free group of rank 3 as a subgroup.*

(1) Explain how to convert a word in a, b, c and their inverses into a word in x, y and their inverses and give an example to show that a reduced word in a, b, c, may not be a reduced word after we convert to x, y.

(2) Consider the word $c^{-1}ac^{-1}b$. Is it equal to the identity in F? Why or why not?

(3) Notice that each of a, b, c and their inverses can be written in the form $\alpha_i\beta_i$. Consider cases to show that if we have
$$\alpha_{i-1}\beta_{i-1}\alpha_i\beta_i\alpha_{i+1}\beta_{i+1}$$
as part of some word, and if $\beta_i = \alpha_{i+1}^{-1}$, then we have only a limited amount of cancellation. (Explain!)

(4) Show that a freely reduced word in a, b, c and their inverses of length n becomes a freely reduced word of length at least n (not $2n$) when written in terms of x, y. This completes the proof of the theorem.

(5) Prove that H is the kernel of the homomorphism $F \to \mathbb{Z}/2\mathbb{Z}$ with $x, y \mapsto 1$. In particular, it has index 2 and is normal. It is called the **even subgroup** of F.

2. An interesting subgroup of D_∞.

Recall that D_∞ is generated by $A(x) = -x$ and $B(x) = 1-x$ as isometries of $(\mathbb{R}, \mathbb{Z})$. Let $AB = B \circ A$, etc.

(1) Let $H < D_\infty$ be the subgroup generated by A and $C = BAB$. Show that H is a subgroup of index 2.

(2) Construct a homomorphism $\phi\colon H \to D_\infty$ as follows. Send A to A and C to B and extend over words. Show that ϕ is a well-defined homomorphism.

(3) Show that ϕ is injective and thus that H and D_∞ are isomorphic.