Class Work 3: Generators and Relations

This is to be completed together as a class, to establish shared notation and terminology.

(1) (Intersection of normal subgroups is normal) Suppose that G is a group and that $\mathcal{H}$ is a nonempty set such that each $H \in \mathcal{H}$ is a normal subgroup of G. Prove that $\bigcap_{H \in \mathcal{H}} H$ is a normal subgroup of G.

(2) (Smallest normal subgroup group containing a subset) Suppose that G is a group and that $R \subset G$ (not necessarily a subgroup, just a subset). Prove that there is a normal subgroup $N \lhd G$ such that $R \subset N$ and for every normal subgroup $H \lhd G$ such that $R \subset H$, then $N \subset H$. We denote N by $\ll R \gg$ and call it the "normal closure of R."

(3) Prove that for any group G, there exists a free group $F(S)$ and a subset $R \subset F$ such that G is isomorphic to $F/\ll R \gg$. We denote the group $F/\ll R \gg$ by $\langle S|R \rangle$ and call it a "presentation" for G. The set S is a set of *generators* for G and R is a set of *relations* for G.

(4) Generate some examples of group presentations. Can you identify the groups in a way other than your presentation?