Topics in Algebra Scott Taylor

Class Work 2: Free Groups

This is to be completed together as a class, to establish shared notation and terminology.

Let $\mathcal{A}$ be a nonempty set, whose elements we call "letters" or "symbols" such that for each symbol $s \in \mathcal{A}$, the symbol $s^{-1} \notin \mathcal{A}$. We let $\mathcal{A}^{-1}$ denote the set whose elements are the symbols s^{-1} for every $s \in \mathcal{A}$. Note that $\mathcal{A} \cap \mathcal{A}^{-1} = \varnothing$. Let $\mathcal{S} = \mathcal{A} \cup \mathcal{A}^{-1}$. The sets $\mathcal{A}$ and $\mathcal{S}$ are called **alphabets**. We ssume that the symbol $\mathbb{1}$ is not an element of $\mathcal{A}$.

Check-in: If $\mathcal{A} = \{a, b\}$, what is $\mathcal{A}^{-1}$?

Let $\mathcal{W} = \mathcal{W}(\mathcal{S})$ be the set whose elements are finite sequences of letters in $\mathcal{S}$, including the "empty sequence" which we denote by $\mathbb{1}$. The set $\mathcal{W}$ is called the set of **words** in the alphabet $\mathcal{S}$. The set of words is also denoted $\mathcal{S}^*$. For $s \in \mathcal{S}$, we define the symbol $(s^{-1})^{-1}$ to be equal to s.

(1) Let $\mathcal{A} = \{a, b, c\}$ and write down some elements of $\mathcal{S}^* = \mathcal{W}$. What is the cardinality of $\mathcal{W}$?

(2) Define multiplication in $\mathcal{W}$ by concatenating words. Write down some examples of concatenation.

(3) For the alphabet $\mathcal{A} = \{a, b\}$, explain why in $\mathcal{W}$ it is **not** true that $abb^{-1}a = aa$.

(4) Explain why multiplication in $\mathcal{W}$ is associative and why $\mathbb{1}$ is the identity for multiplication.

(5) Define a relation on $\mathcal{W}$ by declaring: $w \sim w'$ if there exist $a, b \in \mathcal{W}$ and $x \in S$ so that:
$$w = axx^{-1}b \text{ and } w' = ab \text{ or vice versa.}$$
Prove that $\sim$ is an equivalence relation.

(6) A word w is **reduced** if there does not exist $a, b \in \mathcal{W}$ and $x \in \mathcal{S}$ with $w = axx^{-1}b$. Prove that given $w \in \mathcal{W}$, there exists a unique reduced word $u \in \mathcal{W}$ such that $w \sim u$.

(7) For $w \in \mathcal{W}$, for now let $[w]$ denote its equivalence class. Let $F(\mathcal{A})$ denote the set of equivalence classes of words. Define
$$[w][w'] = [ww']$$
Prove that this is a well defined operation that makes $F(\mathcal{A})$ into a group. It is called **the free group on $\mathcal{A}$**.